Contact: `gwmelvin@colby.edu`

September 25 Summary

Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.4

Keywords: *alternating series, Alternating Series Test, absolutely convergent, conditionally convergent*

Alternating Series Test; Absolute/conditional Convergence

• **Observe:** $\sum \frac{1}{k}$ diverges but $\sum \frac{(-1)^{k+1}}{k}$ converges; $\sum \frac{1}{k^2}$ converges and $\sum \frac{(-1)^{k+1}}{k^2}$ converges.
• Let $\sum a_k$ be a series; here a_k can be **arbitrary**.

- If $\sum |a_k|$ converges then we say that $\sum a_k$ is **absolutely convergent**.
- If $\sum |a_k|$ does not converge but $\sum a_k$ converges then we say that $\sum a_k$ is **conditionally convergent**.

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1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is conditionally convergent: the series $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge while the original series is convergent.
2. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+1}$ is absolutely convergent: the series $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ is convergent.
3. $\sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!}$ is absolutely convergent: the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ is convergent.

• **Note:** if a series $\sum a_k$ converges absolutely then, at the moment, we do not know anything about the convergence of $\sum a_k$. However, it's not too difficult to show the following:

AC $\Longrightarrow$ C

Let $\sum a_k$ be an absolutely convergent series. Then, $\sum a_k$ is convergent.

Example:

1. Consider the series $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$. The terms of this series are neither non-negative nor alternating: we can't apply any of our previous tests for convergence. Let $a_k = \frac{\sin(k)}{k^2}$. Then, for every k,

$$|a_k| = \frac{|\sin(k)|}{k^2} \leq \frac{1}{k^2}$$

Here we use the fact that $|\sin(x)| \leq 1$, for all x. Hence, $\sum |a_k|$ converges by Comparison with the convergent p-series $\sum \frac{1}{k^2}$. Hence, the series $\sum a_k$ converges absolutely, and $\sum a_k$ converges, by AC $\Longrightarrow$ C.

2. Consider the series $\sum_{k=1}^{\infty} \frac{\cos(k)}{7^k + 2^k + \sin^2(k)}$. Let $a_k = \frac{\cos(k)}{7^k+2^k+\sin^2(k)}$. Then, for every k,

$$|a_k| = |\frac{|\cos(k)|}{7^k + 2^k + \sin^2(k)} \leq \frac{1}{7^k + 2^k + \sin^2(k)} < \frac{1}{7^k}$$

The last inequality follows because $2^k + \sin^2(k) > 0$, for all $k \geq 1$. Hence, $\sum |a_k|$ converges by Comparison with the convergent geometric series $\sum 7^{-k}$. Therefore, the series $\sum a_k$ is absolutely convergent and $\sum a_k$ is convergent, by AC $\Longrightarrow$ C.

• **Remark:** Absolutely convergent series behave like 'infinite sums' in the following way: we may rearrange their terms without affecting their convergence behaviour. The same is not true for conditionally convergent series, as we will now see.

Consider the conditionally convergent alternating Harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$: this series converges to a limit L, and $\frac{1}{2} < L < 1$.

$$L = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots \tag{A}$$

Then,

$$\frac{L}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \dots \tag{B}$$

Now, we add $(A) + (B)$

$$\frac{3L}{2} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \dots$$

Suppose we are allowed to rearrange the terms of this last series: then we can see that

$$\frac{3L}{2} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \dots = L \Longrightarrow \quad L = 0$$

This is absurd as $\frac{1}{2} < L < 1$. Therefore, our process of *rearrangement affects the limit*. In fact, even weirder things can happen.

• **Riemann Rearrangement Theorem:** Let $\sum a_k$ be a conditionally convergent series and r be any real number. Then, there is a rearrangement

$$b_1, b_2, b_3, b_4, \dots$$

of the sequence of terms $a_1, a_2, a_3, a_4, \dots$ so that $\sum b_j = r$.