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Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.4

Keywords: *alternating series*, *Alternating Series Test*,

Alternating Series Test

- A series of the form

$$b_1 - b_2 + b_3 - b_4 + \ldots \quad \text{or} \quad -b_1 + b_2 - b_3 + b_4 - \ldots$$

with $b_k > 0$, is called an **alternating series**.

Example:

1. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is alternating.

2. $\displaystyle\sum_{k=1}^{\infty}\sin(k) = \sin(1) + \sin(2) + \sin(3) + \ldots$ is not alternating: $\sin(1), \sin(2) > 0$.

Alternating Series Test (AST)

Let $\sum(-1)^k b_k$ be an alternating series, $b_k > 0$. Suppose

- $b_1 > b_2 > b_3 > b_4 > \ldots > b_k > \ldots$, and
- $\lim b_k = 0$

Then, $\sum(-1)^k b_k$ is convergent. Moreover, in this case $S = \sum(-1)^k b_k$ lies between any two successive partial sums s_n, s_{n+1} i.e.

$$s_n < s < s_{n+1} \quad \text{or} \quad s_{n+1} < s < s_n$$

In particular, $|s - s_n| < b_{n+1}$, for any n.

Example:

1. The series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2+1}$ is convergent: let $b_k = \frac{1}{k^2+1}$. Then,

   - $b_{k+1} = \frac{1}{(k+1)^2+1} < \frac{1}{k^2+1} = b_k$, for all k,
   - $\lim \frac{1}{k^2+1} = 0$.

   Hence, the series is convergent by the Alternating Series Test.

2. The series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \dots$ is convergent: let $b_k = \frac{1}{k!}$. Then,

   - $b_{k+1} = \frac{1}{(k+1)!} < \frac{1}{k!} = b_k$, for all k,
   - $\lim \frac{1}{k!} = 0$.

   Hence, the series converges by the Alternating Series Test. Moreover, if $s = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ then s_{10} approximates s to within $|s - s_{10}| < \frac{1}{11!} \approx 0.000000025$. In fact, $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e}$.

3. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k+2}$: let $b_k = \frac{k}{k+2}$. In this case, $\lim b_k = 1 \neq 0$ so can't apply AST. However, we observe that $\lim (-1)^{k+1} \frac{k}{k+2}$ does not exist, so the series is divergent, by the Test for Divergence.

- **Caution:** if an alternating series fails the first condition of AST then the series need not be divergent: consider the series $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$, where

$$b_k = \begin{cases} \frac{1}{2^k}, & k \text{ odd}, \\ \frac{1}{3^k}, & k \text{ even}. \end{cases}$$

Then, b_k is not decreasing: the sequence is $1/2, 1/9, 1/8, 1/81, 1/32, \dots$. However,

$$\sum_{k=1}^{\infty} b_k = \frac{1}{2} + \frac{1}{9} + \frac{1}{8} + \frac{1}{81} + \frac{1}{32} + \dots$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{4^k} + \sum_{j=1}^{\infty} \frac{1}{4^j}$$

Since $\sum b_k$ is convergent, the series $\sum (-1)^{k+1} b_k$ is absolutely convergent and, therefore, convergent.