Math 122C: Series & Multivariable Calculus
Fall 2018

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Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.3

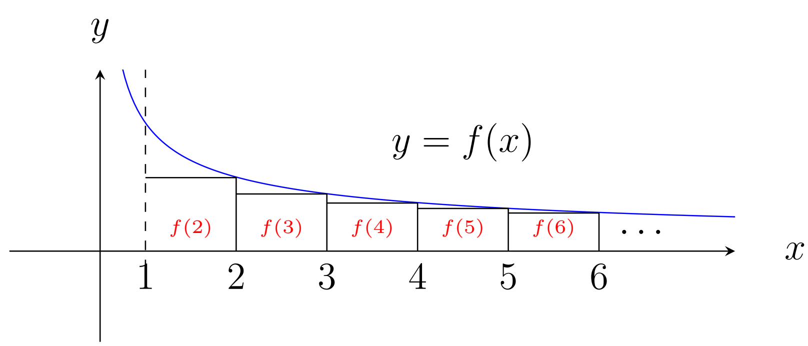
Keywords: *integral test, ratio test, alternating series*

Integral Test, Ratio Test; Alternating Series

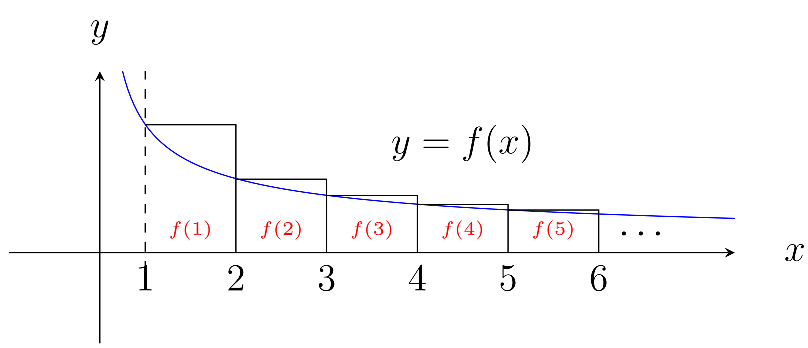
- **p-series Test:** The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is
- **convergent** if $p > 1$,
- **divergent** if $p \leq 1$.

To determine convergence of $\sum_{k=1}^{\infty} \frac{1}{k^p}$ for $1 < p < 2$ we compared the series with the improper integral $\int_1^{\infty} \frac{1}{x^p} dx$. This idea generalises:

Integral Test: let $f(x)$ be **continuous, decreasing, positive** and **defined for** $x \geq 1$.



$$\int_1^{\infty} f(x)dx \quad \text{convergent} \quad \Longrightarrow \sum_{k=1}^{\infty} a_k \quad \text{convergent, where } a_k = f(k)$$



$$\int_1^\infty f(x)dx \quad \text{divergent} \quad \Longrightarrow \sum_{k=1}^{\infty} a_k \quad \text{divergent, where } a_k = f(k)$$

Example: Consider the series $\displaystyle\sum_{k=1}^{\infty} e^{-k}$. We can use the integral test: let $f(x) = e^{-x}$. Then

$$\int_1^\infty e^{-x}dx = \lim_{a\to\infty} \int_1^a e^{-x}dx = \lim_{a\to\infty} \left[-e^{-x}\right]_1^a = \lim_{a\to\infty} \left(e^{-1} - e^{-a}\right) = e^{-1}$$

Hence, since $\int_1^\infty e^{-x}dx$ convergent so is $\displaystyle\sum_{k=1}^{\infty} e^{-k}$, by the Integral Test.

• **Ratio Test:** Let $\sum a_k$ be a series of positive terms, $a_k > 0$ for all $k = 1, 2, 3, \ldots$. Let $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.

- $L < 1$: then $\sum a_k$ convergent;
- $L > 1$: then $\sum a_k$ divergent;
- $L = 1$: further testing required.

Example:

1. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$, $a_k = \frac{1}{k!}$

   $$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)!} \cdot \frac{k!}{1} = \frac{k!}{(k+1)!} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}{1 \cdot 2 \cdot 4 \cdot \cdots \cdot k \cdot (k+1)} = \frac{1}{k+1} \to 0, \text{ as } k \to \infty$$

   Hence, $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 0 < 1$ and the series is convergent, by the Ratio Test.

2. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{k^2}$, $a_k = \frac{5^k}{k^2}$.

   $$\frac{a_{k+1}}{a_k} = \frac{5^{k+1}}{(k+1)^2}\frac{k^2}{5^k} = \frac{5k^2}{(k+1)^2} = \frac{5}{(1+1/k)^2} \to 5, \text{ as } k \to \infty$$

   Hence, $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 5 > 1$ and the series is divergent, by the Ratio Test.

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, $a_k = \frac{1}{k}$

   $$\frac{a_{k+1}}{a_k} = \frac{1}{k+1} \cdot \frac{k}{1} = \frac{k}{k+1} = \frac{1}{1+1/k} \to 1, \text{ as } k \to \infty$$

   Hence, further testing is required. Similarly, for the series $\sum \frac{1}{k^2}$ the Ratio Test is **inconclusive**.

• **Remark:** Ratio Test is good to try if a_k is an expression involving exponential terms (e.g. 3^k etc.) or factorials (e.g. $k!$).

• **Alternating Series:** A series is **alternating** if it takes the form

$$b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots$$

or

$$-b_1 + b_2 - b_3 + b_4 - b_5 + b_6 - \dots$$

with $b_k > 0$

• The **alternating Harmonic Series** is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$