Math 122C: Series & Multivariable Calculus
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Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.3

Keywords: *p-series, p-series test*

p-series Test

• **Recall:**

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots \qquad \text{is divergent when } p \leq 1$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots \qquad \text{is convergent.}$$

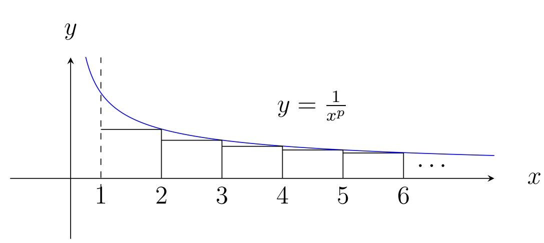
• **Observe:** For $p \geq 2$, and any $k = 1, 2, 3, \dots$, $\frac{1}{k^p} \leq \frac{1}{k^2}$. Hence, by Comparison Test, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent whenever $p \geq 2$.

• **What happens for** $1 < p < 2$**?** We will determine this behaviour by comparing the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $1 < p < 2$, with an appropriate improper integral.

• Fix $1 < p < 2$. Let $f(x) = \frac{1}{x^p}$, $x > 0$. Then,

$$\begin{aligned}
\int_1^{\infty} f(x)dx &= \lim_{a \to \infty} \int_1^a \frac{1}{x^p} dx \\
&= \lim_{a \to \infty} \left[\frac{1}{1-p} \frac{1}{x^{p-1}} \right]_1^a \\
&= \lim_{a \to \infty} \left(\frac{1}{1-p} \left(\frac{1}{a^{p-1}} - 1 \right) \right) \\
&= \frac{1}{1-p} \left(\lim_{a \to \infty} \left(\frac{1}{a^{p-1}} - 1 \right) \right) = \frac{1}{p-1}
\end{aligned}$$

The last equality follows because $p - 1 > 0$ (i.e. $p > 1$) so that, as a gets very large, $\frac{1}{a^{p-1}} \to 0$. In particular, the improper integral $\int_1^{\infty} f(x)dx$ converges and its value $(= \frac{1}{p-1})$ computes the area below the graph $y = f(x)$, $1 \leq x < \infty$.



The rectangles above have successive areas $\frac{1}{2^p}, \frac{1}{3^p}, \frac{1}{4^p}, \frac{1}{5^p}, \ldots$. Therefore,

$$\sum_{k=2}^{\infty} \frac{1}{k^p} = \text{ combined area of all rectangles} < \int_1^{\infty} \frac{1}{x^p} dx$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \sum_{k=2}^{\infty} \frac{1}{k^p}$$

converges whenever $1 < p < 2$.

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p-series Test

The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is

- **convergent** if $p > 1$,
- **divergent** if $p \leq 1$.

Example:

1. $\sum_{k=1}^{\infty} \frac{1}{k^{3/2} + 5k + 1}$: for each $k = 1, 2, 3, \ldots$ $\frac{1}{k^{3/2} + 5k + 1} < \frac{1}{k^{3/2}}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is convergent, by p-series Test, the series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2} + 5k + 1}$ is convergent, by Comparison Test.

2. $\sum_{k=1}^{\infty} \frac{1}{5k + 1}$: we might think to compare with the divergent series $\sum_{k=1}^{\infty} \frac{1}{5k}$. However, $\frac{1}{5k+1} < \frac{1}{5k}$, for each k, and the Comparison Test does not apply. All is not lost: note that, for every $k = 1, 2, 3, \ldots$,

$$5k + 1 < 10k \quad \Longrightarrow \quad \frac{1}{10k} < \frac{1}{5k + 1}$$

Moreover, the series $\sum_{k=1}^{\infty} \frac{1}{10k}$ is divergent, so that the series $\sum_{k=1}^{\infty} \frac{1}{5k + 1}$ is divergent, by the Comparison Test.

• **Remark:** In the second example we have used the following FACT: Let c be a constant. Then

$$\sum a_k \text{ convergent/divergent } \Leftrightarrow \sum c a_k \text{ convergent/divergent}$$