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September 18 Summary

Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.3

Keywords: *comparison test*, *p-series*, *p-series test*

Comparison Test; *p*-series

• **Recall:** the Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is **divergent**.

• To obtain this result we *compared* the Harmonic series with another series. Today we will develop this idea of *comparison*.

Consider the series $\sum_{k=0}^{\infty} \frac{1}{5^k + 2}$. This series looks like a geometric series but we can't apply the Geometric Series Theorem (because it's not a geometric series). Note

$$\frac{1}{5^k + 2} < \frac{1}{5^k}, \quad k = 0, 1, 2, \dots$$

$$\implies \frac{1}{5^0 + 2} + \frac{1}{5^1 + 2} + \dots + \frac{1}{5^n + 2} < \frac{1}{5^0} + \frac{1}{5^1} + \dots + \frac{1}{5^n} < \sum_{k=0}^{\infty} \frac{1}{5^k} = \frac{1}{1 - 1/5} = \frac{5}{4}$$

The sum on the left is s_n, the n^{th} partial sum of $\sum_{k=0}^{\infty} \frac{1}{5^k + 2}$, the sum on the right is the n^{th} partial sum of $\sum_{k=0}^{\infty} \frac{1}{5^k}$.

• **Observation:** $\{s_n\}$ is bounded above (by 5/4); $\{s_n\}$ is increasing - $s_{n+1} = s_n + \frac{1}{5^{n+1}+2} > s_n$. Hence, the sequence of partial sums $\{s_n\}$ is convergent $\implies \sum_{k=0}^{\infty} \frac{1}{5^k + 2}$ is convergent.

The above example generalises to the following result.

> **Comparison Test:**
>
> Let $\sum a_k$ and $\sum b_k$ be series, and suppose that $0 \le a_k \le b_k$ for every k.
>
> - If $\sum b_k$ converges then $\sum a_k$ converges and $\sum a_k \le \sum b_k$;
> - If $\sum a_k$ diverges then $\sum b_k$ diverges.

• To take advantage of the Comparison Test we need to build a bank of standard series whose behaviour is known - these are the series we will use to compare other series against.

Example:

1. The series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Note,

$$\frac{1}{\sqrt{k}} \leq \frac{1}{k}, \quad \text{for every } k = 1, 2, 3, \ldots$$

   Hence, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, by the Comparison Test.

   Observation: This example generalises: if $p \leq 1$ then $\frac{1}{k^p} \leq \frac{1}{k}$, for every $k = 1, 2, 3, \ldots$. Hence, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges, for any $p \leq 1$, by the Comparison Test.

2. Recall the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots = 1$. We can rewrite this series as

$$\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots = 1$$

   For each $k = 2, 3, 4, \ldots$, $\frac{1}{k(k-1)} < \frac{1}{k^2}$. Hence, the series $\sum_{k=2}^{\infty} \frac{1}{k^2}$ is convergent and $\sum_{k=2}^{\infty} \frac{1}{k^2} \leq \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1$.

   This implies the series $1 + \sum_{k=2}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent and $\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$. In fact, in the 1730s Euler showed that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.645$$