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Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.2

Keywords: *Test for Divergence, Harmonic series*

Test for Divergence; Harmonic series

• Let $\sum_{k=1}^{\infty} a_k$ be a **convergent** series with partial sums $s_n = a_1 + \ldots + a_n$. Observe that $a_n = s_n - s_{n-1}$, $n = 2, 3, \ldots$. Since $\sum_{k=1}^{\infty} a_k$ convergent, $\lim s_n = s$ exists. **Note:** $\lim s_{n-1} = s$ too. Hence,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (s_n - s_{n-1}) \\ &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= s - s = 0\end{aligned}$$

• **Summary:** if $\sum_{k=1}^{\infty} a_k$ is convergent then $\{a_k\}$ is convergent and $\lim_{k\to\infty} a_k = 0$.

• **Equivalently:** (**Test for Divergence**) If either $\{a_k\}$ is divergent or $\lim a_k \neq 0$ then $\sum a_k$ is divergent.

Example:

1. $\sum_{k=1}^{\infty} \sin(k)$: $a_k = \sin(k)$. The sequence $\{a_k\}$ is divergent. Hence, the series $\sum_{k=1}^{\infty} \sin(k)$ is divergent by the Test for Divergence.

2. $\sum_{k=1}^{\infty} \frac{3k+2}{5k-1}$, $a_k = \frac{3k+2}{5k-1}$. Using Limit Laws for Sequences, can show

$$\lim a_k = \lim \frac{3k+2}{5k-1} = \lim \frac{3+2/k}{5-1/k} = \frac{3+0}{5-0} = \frac{3}{5} \neq 0$$

Since $\lim a_k \neq 0$, the series $\sum_{k=1}^{\infty} a_k$ is divergent, by the Test for Divergence.

• **Careful:** Test for Divergence is not saying: if $\lim a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ is convergent.

The following example is fundamental.

• The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the **Harmonic Series**. Note: $\lim \frac{1}{k} = 0$. However, the Harmonic series is **divergent**: suppose the series were convergent with limit H. Then,

$$\begin{aligned} H = \sum_{k=1}^{\infty} \frac{1}{k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \dots \\ &= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \dots \\ &= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\ &= \frac{1}{2} + H \end{aligned}$$

In summary:

$$H > \frac{1}{2} + H$$

This inequality is nonsense. Therefore, our assumption that the Harmonic Series is convergent can't possibly be true. The only remaining possibility, therefore, is that the Harmonic Series is divergent.