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September 14 Summary

Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.2

Keywords: *geometric series, Test for Divergence*

Geometric Series Theorem; Test for Divergence

- Let $x \neq 1$ be a real number. Then, $\sum_{k=0}^{\infty} x^k$ is a **geometric series** with partial sum

$$s_n = 1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

In HW1 you showed that

$$\lim_{n\to\infty} x^{n+1} = \begin{cases} 0, & |x| < 1, \\ \text{DNE}, & x < -1 \text{ or } x > 1 \end{cases} \implies \lim_{n\to\infty} s_n = \begin{cases} \frac{1}{1-x}, & |x| < 1, \\ \text{DNE}, & x < -1 \text{ or } x > 1 \end{cases}$$

Hence, $\sum_{k=0}^{\infty} x^k \begin{cases} \text{converges}, & -1 < x < 1, \\ \text{diverges}, & -1 < x \text{ or } x > 1 \end{cases}$ and

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad -1 < x < 1$$

This conclusion is the **Geometric Series Theorem**.

- If $x = \pm 1$ then $\sum_{k=0}^{\infty} x^k$ is divergent:
  - the case $x = -1$ was seen last lecture,
  - the case $x = 1$: the partial sum is $s_n = 1 + 1 + 1 + \ldots + 1 = n$. Since $\{s_n\}$ is divergent, the series $\sum_{k=0}^{\infty} 1$ is divergent.

Example:

1. $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$ is geometric series with $x = \frac{2}{3}$. Since $-1 < \frac{2}{3} < 1$, the series converges (by the Geometric Series Theorem), and $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1 - 2/3} = 3$
2. $\sum_{k=0}^{\infty} \frac{5^k}{3^k}$ diverges: this is a geometric series with $x = \frac{5}{3} > 1$.

3. Consider the series $\sum_{k=0}^{\infty} \frac{3^k}{2^{2k}}$. Rewrite $\frac{3^k}{2^{2k}} = \frac{3^k}{(2^2)^k} = \left(\frac{3}{4}\right)^k$. Hence, we have geometric series with $x = \frac{3}{4}$ so that the series is convergent and $\sum_{k=0}^{\infty} \frac{3^k}{2^{2k}} = \frac{1}{1-3/4} = 4$.

4. Consider the decimal expansion $0.999\ldots$. Can write using a geometric series:

$$
\begin{aligned}
0.999\ldots &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots \\
&= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right) \\
&= \frac{9}{10}\sum_{k=0}^{\infty} \frac{1}{10^k} \\
&= \frac{9}{10}\left(\frac{1}{1-1/10}\right) = 1
\end{aligned}
$$

Limit Laws for Series (Theorem 5, Section 11.2)

• Let $\sum a_k$ and $\sum b_k$ be convergent series, $\sum a_k = A$, $\sum b_k = B$. Then, $\sum(a_k+b_k) = A + B$ is convergent.

• If c is a constant then $\sum(ca_k) = cA$.