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September 12 Summary

Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.2

Keywords: *infinite series, convergence of series, sequence of partial sums, geometric series*

Infinite Series; Geometric Series

- An **infinite series** is an expression

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_k + \ldots$$

A series **converges** if its **sequence of partial sums** $\{s_n\}$ converges; $s_n = a_1 + a_2 + \ldots + a_n$. If $\lim_{n\to\infty} s_n = s$ (s finite real number) then write $\sum_{k=1}^{\infty} a_k = s$. Otherwise, say $\sum_{k=1}^{\infty} a_k$ **diverges**.

Example:

1. $\sum_{k=1}^{\infty} k = 1 + 2 + 3 + \ldots$

   Sequence of partial sums:

   $$s_1 = 1, \quad s_2 = 1+2 = 3, \quad s_3 = 1+2+3 = 6, \quad \ldots \quad s_n = 1+2+\ldots+n = \frac{1}{2}n(n+1)$$

   The sequence $\{s_n\}$ is unbounded: $\frac{1}{2}n(n+1) \to +\infty$. Hence, $\sum_{k=1}^{\infty} k$ diverges (to $+\infty$).

2. $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

   Sequence of partial sums:

   $$s_1 = 1/2, \quad s_2 = 1/2+1/4 = 3/4, \quad s_3 = 1/2+1/4+1/8 = 7/8, \quad \ldots \quad s_n = 1-\frac{1}{2^n}$$

   The sequence $\{s_n\}$ is convergent: using Limit Laws

   $$\lim_{n\to\infty} s_n = 1 - 0 = 1$$

   Hence, $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges and $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$

Sequence of partial sums:

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)}$$

Note: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Hence,

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Coloured terms cancel $\implies$ $s_n = 1 - \frac{1}{n+1}$. Hence, $\{s_n\}$ converges and $\lim_{n\to\infty} s_n = 1 - 0 = 1$. Therefore, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

4. $\displaystyle\sum_{k=1}^{\infty} (-1)^k = -1 + 1 + (-1) + 1 + (-1) + \dots$

Sequence of partial sums:

$$s_n = \begin{cases} -1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

The sequence $\{s_n\}$ does not converge; hence, the series $\displaystyle\sum_{k=1}^{\infty} (-1)^k$ is divergent.

Geometric Series: Let x be a real number. Consider the series

$$\sum_{k=1}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

Call such a series a **geometric series**. Sequence of partial sums $\{s_n\}$:

$$s_n = 1 + x + x^2 + \dots + x^n$$

TRICK:

$$\begin{aligned} (1-x)s_n &= (1-x)(1 + x + x^2 + \dots + x^n) \\ &= 1 + x + x^2 + x^3 + \dots + x^n \\ &\qquad - x - x^2 - x^3 - \dots - x^n - x^{n+1} \\ &= 1 - x^{n+1} \end{aligned}$$

If $x \neq 1$ then $s_n = \dfrac{1 - x^{n+1}}{1 - x}$. HW1 Problem 18 gives

$$\lim_{n\to\infty} x^{n+1} = \begin{cases} 0, & |x| < 1, \\ \text{DNE}, & x < -1 \text{ or } x > 1 \end{cases} \qquad \implies \qquad \lim_{n\to\infty} s_n = \begin{cases} \frac{1}{1-x}, & |x| < 1, \\ \text{DNE}, & x < -1 \text{ or } x > 1 \end{cases}$$