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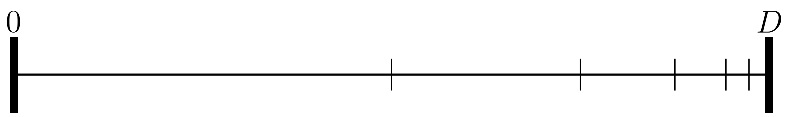
September 11 Summary

Supplementary References:

- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.2

Keywords: *infinite series, convergence of series, sequence of partial sums*

Infinite Series; Sequence of Partial Sums



- **Last time:** Defined a sequence $\{s_n\}$, where

$$s_n = \frac{D}{2} + \frac{D}{4} + \frac{D}{8} + \ldots + \frac{D}{2^n}$$

and

$$s_n = s_{n-1} + \frac{D}{2^n}$$

s_n is the distance covered after n steps across a room (having width D) where, at the n^{th} step, we half the distance to the opposite side of the room. • $\{s_n\}$ is **nondecreasing** and **bounded above**, so it converges, by MBT; $\lim_{n\to\infty} s_n = D$

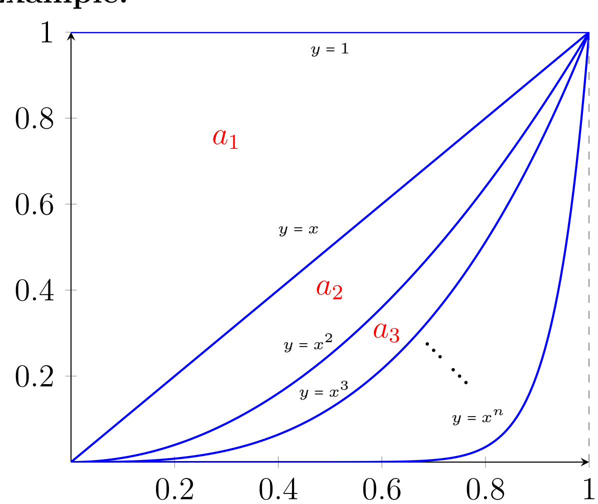
- Seems reasonable to write

$$\frac{D}{2} + \frac{D}{4} + \frac{D}{8} + \frac{D}{16} + \ldots + \frac{D}{2^n} + \frac{D}{2^{n+1}} + \ldots = D$$

despite fact that it's impossible to 'sum' an infinite collection of numbers. In particular, if $D = 1$ then

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots = 1$$

- **Example:**



Consider graphs $y = x^n$, $n = 0, 1, 2, 3, \ldots$

Let a_n = area between $y = x^{n-1}$ and $y = x^n$, $0 \le x \le 1$

Define

$$s_n = \text{ area lying above } y = x^n \text{ in unit square} = a_1 + a_2 + a_3 + \ldots + a_n$$

- Compute: $a_n = \int_0^1 (x^{n-1} - x^n)dx = \frac{1}{n(n+1)}$

$$\implies \quad s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)}$$

$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)}$$

- $\{s_n\}$ is **nondecreasing** and **bounded above**. By MBT, $\{s_n\}$ converges; $\lim_{n\to\infty} s_n = 1$ (the limit is the total area of unit square) • As before, seems reasonable to write:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots = 1$$

- **Observe:** we have just written down two *infinite sums* but we must be careful with our interpretation: it's literally impossible to 'add' an infinite collection of numbers (not enough time). What the 'infinite sum' *means* is that we have taken a limit of a sequence whose terms are *finite sums.*

- An **infinite series** (or **series**) is an expression

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$$

where $a_1, a_2, a_3, \ldots$ is a sequence of real numbers.

Example:

1. $\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \ldots$
2. $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$
3. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots$
4. $\sum_{k=1}^{\infty} \frac{1}{k!} = \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$

Aim: determine whether a series $\sum_{k=1}^{\infty} a_k$ is *meaningful* i.e. does it **converge** to a finite real number? *need to make this notion rigorous.*

- The n^{th} **partial sum** of a series $\sum_{k=1}^{\infty} a_k$ is the (finite number)

$$s_n = a_1 + a_2 + \ldots + a_n$$

Say the series $\sum_{k=1}^{\infty} a_k$ **converges** if the sequence of partial sums $\{s_n\}$ converges i.e. $S = \lim_{n\to\infty} s_n$. In this case, we write $\sum_{k=1}^{\infty} a_k = S$.

- **Note:** if a series converges then we have (by definition) $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

• **Recall:** (**Sigma Notation**) Given a finite collection of numbers $a_1, \ldots, a_n$ we denote

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_k$$

The symbol $\sum$ is the Greek capital letter *sigma*.