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September 10 Summary

Supplementary References:

- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.1-2

Keywords: *Monotonic Bounded Theorem, series*

Monotonic Bounded Theorem; Series

• **Recall:** a sequence is a list of real numbers

$$a_1, a_2, a_3, a_4, \ldots$$

We represent sequences via their graph to better illustrate their behaviour.

• We introduced notion of a sequence $\{a_n\}$ **converging to** L; L is the **limit** of the sequence, write $L = \lim_{n\to\infty} a_n$.

Example:

1. $\lim_{n\to\infty} \frac{a}{n^p} = 0$, for a constant and $p > 0$;
2. We can use limit laws (Theorem 1, Section 11.1)

$$\lim_{n\to\infty} \frac{2n+1}{3n^2+2} = \lim_{n\to\infty} \frac{n(2+1/n)}{n^2(3+2/n^2)} = \lim_{n\to\infty} \left(\frac{1}{n} \cdot \frac{2+1/n}{3+2/n^2}\right) = \left(\lim_{n\to\infty} \frac{1}{n}\right) \frac{\lim_{n\to\infty}(2+1/n)}{\lim_{n\to\infty}(3+2/n^2)} = 0 \cdot \frac{2}{3} = 0$$

• Squeeze Theorem: use known behaviour of sequences to determine behaviour of a new sequence e.g. $a_n = \frac{\sin(3n^2+1)}{n^3}$,

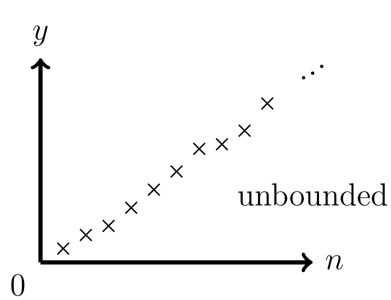
$$-1 \leq \sin(3n^2+1) \leq 1 \quad \Longrightarrow \quad -\frac{1}{n^3} \leq \frac{\sin(3n^2+1)}{n^3} \leq \frac{1}{n^3}$$

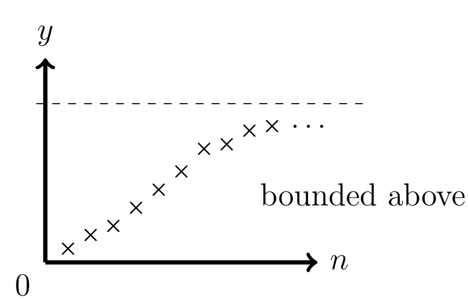
Squueeeeeeze: $\lim -\frac{1}{n^3} = \lim \frac{1}{n^3} = 0$, hence $\lim a_n = 0$, by Squeeze Theorem.

Determining convergence is a two-step process: (1) show that a sequence converges; (2) determine the limit. Sometimes it's sufficient to show that a sequence converges without finding its limit.

• **Observe:** if a sequence $\{a_n\}$ is nondecreasing then two possibilities

1. $\{a_n\}$ unbounded above,
2. $\{a_n\}$ bounded above.





In this case, the sequence $\{a_n\}$ is

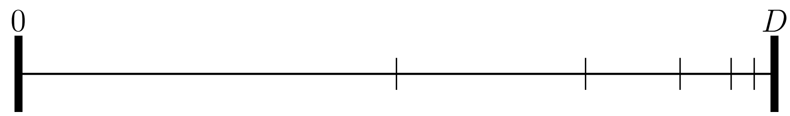
1. divergent (to $+\infty$),
2. convergent.

• **Monotonic Bounded Theorem** (MBT)

If $\{a_n\}$ is $\begin{cases} \text{nondecreasing} \\ \text{nonincreasing} \end{cases}$ and $\begin{cases} \text{bounded above} \\ \text{bounded below} \end{cases}$ then $\{a_n\}$ is convergent.

• **Remark:** MBT only shows that $\lim a_n$ exists but does not specify $\lim a_n$.

Example: (Related to Zeno's Paradox of tortoise and Achilles)



I walk across a room having width D metres as follows:

- (STEP 1) Half distance to opposite side of room
- (STEP 2) Half remaining distance at STEP 1
- (STEP 3) Half remaining distance at STEP 2
- $\vdots$
- (STEP n) Half remaining distance at STEP $n-1$.

Let

$$s_n = \text{ distance covered after STEP } n \text{ (in metres)}$$

Then,

$$s_1 = \frac{D}{2}$$

$$s_2 = \frac{D}{2} + \frac{1}{2}\left(D - \frac{D}{2}\right) = \frac{D}{2} + \frac{D}{4}$$

$$s_3 = \frac{D}{2} + \frac{D}{4} + \frac{1}{2}\left(D - \frac{D}{2} + \frac{D}{4}\right) = \frac{D}{2} + \frac{D}{4} + \frac{D}{8}$$

$$\vdots$$

$$s_n = \frac{D}{2} + \frac{D}{4} + \ldots + \frac{D}{2^{n-1}} + \frac{D}{2^n}$$

The sequence $\{s_n\}$ is

- nondecreasing (s_n is obtained from s_{n-1} by adding on a positive value = half the remaining distance to the opposite side of the room)
- bounded above (an upper bound is D)

Hence, by MBT, the sequence is convergent i.e. if we take an 'infinite' number of steps then we cover a finite distance.