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October 3 Summary

Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.7

Keywords: *Taylor series*

Taylor series

• **Recall:** If $f(x)$ can be represented by a power series on some interval I, $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$, then $a_k = \frac{f^{(k)}(c)}{k!}$.

• **Question:** What functions admit power series representations?
• **Remark:** any function must be infinitely differentiable.
• Approach to question: let $f(x)$ be infinitely differentiable on I.

1. Associate a power series to $f(x)$
2. Check when power series equals $f(x)$.

• Let $f(x)$ be an infinitely differentiable function. Define the **Taylor series associated to $f(x)$ centred at** c to be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots$$

• **Remark:** we are not (yet) saying that the Taylor series equals $f(x)$. However, the Taylor series does equal $f(x)$ at $x = c$.
• The Taylor series associated to $f(x)$ centred at $c = 0$ is also called a **Maclaurin series**.

Example:

1. Let $f(x) = \sin(x)$, $c = 0$. The Maclaurin series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

We compute

$$f(x) = \sin(x) \implies f(0) = 0$$
$$f'(x) = \cos(x) \implies f'(0) = 1$$
$$f''(x) = -\sin(x) \implies f''(0) = 0$$
$$f'''(x) = -\cos(x) \implies f'''(0) = -1$$
$$f''''(x) = \sin(x) \implies f''''(0) = 0$$

$$\vdots$$

Then, Maclaurin series is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$$

The Maclaurin series converges for all x.

2. $f(x) = \ln(x)$, $c = 1$. The Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = f(1) + f'(1)(x-1)^k + \frac{f''(1)}{2!}(x-1)^2 + \ldots$$

Compute

$$f^{(k)}(x) = \begin{cases} \frac{(k-1)!}{x^k}, & k \text{ odd} \\ -\frac{(k-1)!}{x^k}, & k \text{ even} \end{cases} \implies f^{(k)}(1) = \begin{cases} (k-1)!, & k \text{ odd} \\ -(k-1)!, & k \text{ even} \end{cases}$$

Hence, Taylor series is

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-1)!}{k!}(x-1)^k$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

The Taylor series converges for $|x-1| < 1$.

- **Remark:** If let we let $x = y + 1$ then the above Taylor series becomes

$$y - \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} - \ldots$$

We've seen that this series equals $\ln(1+y)$, whenever $|y| < 1$. Hence, for all $|x-1| < 1$ we have

$$\ln(x) = \ln(y+1) = y - \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} - \ldots = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

That is, for $|x-1| < 1$, **the Taylor series of $f(x) = \ln(x)$ centred at $c = 1$ equals $f(x)$!**

Some Maclaurin Series

- $f(x) = \sin(x)$, $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
- $f(x) = \cos(x)$, $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
- $f(x) = e^x$, $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

The above series converge for all x. We've already seen that the Maclaurin Series for $f(x) = e^x$ is equal to $f(x)$.

Fact: Each of the above functions equals its Maclaurin series, for all x.