Colby

Math 122C: Series & Multivariable Calculus
Fall 2018

Contact: `gwmelvin@colby.edu`

October 2 Summary

Supplementary References:

- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.6

Keywords: *power series, representing functions as power series*

Properties of Power Series

• **Recall:** A function $f(x) = \displaystyle\sum_{k=0}^{\infty} a_k(x-c)^k$, with domain I, has the following properties:

1. $f'(x)$ is obtained by differentiating $f(x)$ term-by-term and domain of $f'(x)$ is I,
2. $\int f(x)dx$ is obtained by integrating term-by-term and domain of $\int f(x)dx$ is I.

Example:

1. $f(x) = \frac{1}{x+x}$, with domain $I = (-1, 1)$. We can write

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

Integrating, we have

$$\ln(1+x) + C = \int f(x)dx = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \dots)dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \quad \text{valid whenever } |x| < 1$$

Substituting $x = 0$ gives $C = 0$. Note

$$\ln(2) = -\ln(1/2) = -\ln(1+(-1/2)) = \frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \frac{1}{4\cdot 16} + \dots = \sum_{k=1}^{\infty} \frac{1}{k\cdot 2^k}$$

Therefore, we've been able to determine the limit of the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\cdot 2^k} = \ln(2)$ using power series representations of functions.

2. Let $f(x) = e^{-x^2} = \sum_{k=0}^{\infty} = \frac{(-x^2)^k}{k!} 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$.

Fact: there is no elementary antiderivative of $f(x)$ - this means it's impossible to find an antiderivative of $f(x)$ that is an expression involving all standard functions in calculus. However, $f(x)$ is continuous and therefore integrable i.e. $\int_0^1 f(x)dx$ can be determined. (Also, an antiderivative of $f(x)$ does exist, by

Fundamental Theorem of Calculus). Using the above power series expansion, we have

$$s = \int_0^1 f(x)dx = \int_0^1 (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots)dx$$
$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots\right]_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1) \cdot k!}$$

This is an alternating series so that, for example, we know $|s - s_{10}| < \frac{1}{23 \cdot 11!} \approx 0.000000001$

• Power series representations also allow us to determine limits without L'Hopital's rule: e.g.

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{1 + x + x^2/2! + \ldots - 1}{x} = \lim_{x \to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} = 1$$

• **Basic Question:** Let $f(x)$ be a function. Is it possible to represent $f(x)$ on some interval using a power series?
• (**First Answer**)

- Not for any function: e.g. $f(x) = |x|$, centre $c = 0$. Can't represent as a power series centred at $c = 0$ since $f'(0)$ DNE.

- (Non-trivial) Define $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$.

  Fact: $f(x)$ is differentiable everywhere but there's no series expansion with centre $c = 0$.

Necessary conditions:

• If $f(x)$ can be represented as a power series on I then $f^{(k)}(x)$ exists for all k and $f^{(k)}(x)$ defined on I.
• If $f(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \ldots$ then

$$f(c) = a_0, \quad f'(c) = a_1, \quad f''(c) = 2a_2, \quad f'''(c) = 3.2a_3$$

$$\Longrightarrow a_k = \frac{f^{(k)}(c)}{k!}, \quad \text{for every } k \geq 0$$