Contact: gwmelvin@colby.edu

October 1 Summary

Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.6

Keywords: *power series, interval of convergence, properties of power series*

Properties of Power Series

- Given a power series

$$a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

one of three situations occur:

(A) power series converges at $x = c$ only: e.g. $\sum_{k=0}^{\infty} k!x^k$.

(B) power series converges on a finite interval of form $(c-r, c+r)$, $[c-r, c+r)$, $(c-r, c+r]$, or $[c-r, c+r]$: e.g. $\sum_{k=0}^{\infty} \frac{(x+2)^k}{k}$ converges on $[-3, -1)$

(C) power series converges **for all** x: e.g. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

We call this (finite/infinite) interval the **interval of convergence**; r is the **radius of convergence**.

Properties of power series as functions
Given a power series having interval of convergence I, we can define the function

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k, \quad x \text{ in } I$$

- **Question:** What properties does the function $f(x)$ possess?

Example: Let $f(x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$, $I = (-1, 1)$. Then,

- $f(x)$ is continuous on I
- $f(x)$ is differentiable on I
- $f(x)$ is integrable on I (i.e. there exists an antiderivative)

- **Theorem:** Let $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ be a function defined by a power series, with interval of convergence I. Then,

1. $f(x)$ is differentiable on I and

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \ldots$$

i.e. to obtain $f'(x)$ we differentiate the series defining $f(x)$ **term-by-term**. Moreover, the interval of convergence of $f'(x)$ is I.

2. $f(x)$ is integrable on I and

$$\int f(x)dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \frac{a_3}{4}(x-c)^4 + \ldots$$

i.e. to obtain $\int f(x)dx$ we integrate the series defining $f(x)$ **term-by-term**. Moreover, the interval of convergence of $\int f(x)dx$ is I.

Example:

1. Let $f(x) = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$, with domain $I = (-1, 1)$. Then

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{k=0}(k+1)x^k$$

and this power series has interval of convergence $I = (-1, 1)$.

2. Let $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, having interval of convergence $(-\infty, \infty)$. Then, differentiating term-by-term

$$f'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = f(x)$$

FACT: Any function satisfying $f'(x) = f(x)$ must be of the form $f(x) = Ae^x$, for some constant A. Since $f(0) = 1$, we have $A = 1$. Hence, $f(x) = e^x$ i.e.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Similarly, $e^{-x} = \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{k!}$.