Colby

Math 122C: Series &
Multivariable Calculus
Fall 2018

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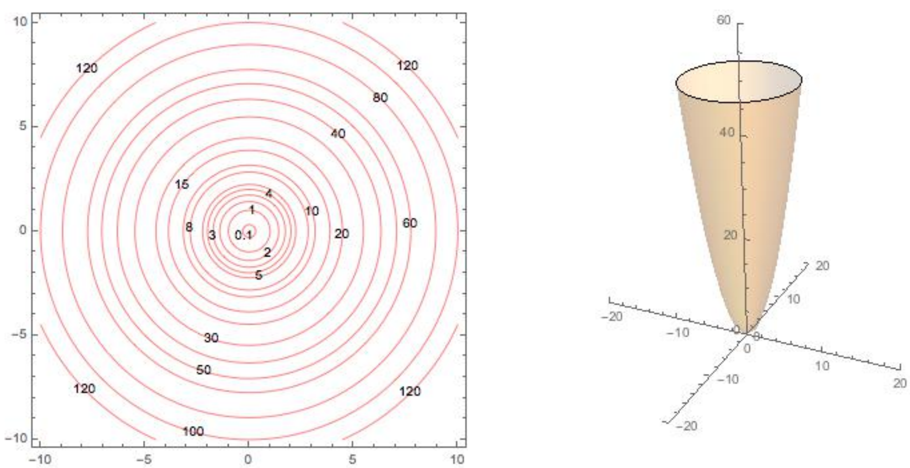
Midterm 2 Partial Review

• A **level curve (having level** c**)** of $f(x, y)$ is the collection of points (x, y) satisfying $f(x, y) = c$. **Level curves are curves in the domain of** $f(x, y)$ i.e. they live in $\mathbb{R}^2$.
A diagram representing several level curves, labelled by level, is called a **level curve diagram**. Level curve diagrams are also called **contour plots**, borrowing terminology from cartography and topographical maps.

• The **graph** of $f(x, y)$ is the collection of all points $(x, y, z) \in \mathbb{R}^3$ satisfying $z = f(x, y)$. **The graph of a function lives in** $\mathbb{R}^3$.

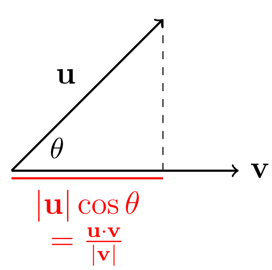


Contour plot (left) and graph (right) of $f(x, y) = x^2 + y^2$.

• The **dot product u · v** is an operation on two vectors **u**, **v** that produces a scalar quantity. The dot product is very useful for geometry because

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

where $0 \leq \theta \leq \pi$ is the angle between **u** and **v**.



The vector parallel to **v** having length $|\mathbf{u}| \cos\theta$, $\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$, is called the **vector projection of u on v**.

• The **cross product** $\mathbf{u} \times \mathbf{v}$ is an operation on two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ that produces a vector quantity. **The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$**. We have

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

The cross product is defined only for vectors in $\mathbb{R}^3$. It is important to remember the geometric properties of the cross product.

• Given two points $P = (a, b, c), Q = (d, e, f) \in \mathbb{R}^3$, the **displacement vector from P to Q** is

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (d, e, f) - (a, b, c) = (d - a, e - b, f - c)$$

• To define a **line** L in $\mathbb{R}^3$ you need a **point on the line** P and a direction vector $\mathbf{u} \in \mathbb{R}^3$ parallel to L:

$$L: \qquad P + t\mathbf{u}, \quad t \in \mathbb{R}$$

For example, the line passing through the two points $P = (1, 2, 1)$ and $Q = (3, 3, 3)$ is found as follows: let $\mathbf{u} = \overrightarrow{PQ} = (2, 1, 2)$. Then, we parameterise the line as

$$\mathbf{r}(t) = (1, 2, 1) + t(2, 1, 2) = (1 + 2t, 2 + t, 1 + 2t), \quad t \in \mathbb{R}$$

You should know how to recognise parameterised lines.

• To define a **plane** Π you need a **point on the plane** P and a **normal vector** $\mathbf{n} = (a, b, c) \in \mathbb{R}^3$. The normal vector is perpendicular to Π; there are (infinitely) many choices for $\mathbf{n}$. The equation of the plane is

$$ax + by + cz = d,$$

where $d = \mathbf{n} \cdot \overrightarrow{OP}$.

For example, to define the plane containing the points $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$ we need only determine a normal vector $\mathbf{n}$. We do this as follows: we take $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = (-1, 1, 0) \times (-1, 0, 1) = (1, 1, 1)$. Then, $d = \mathbf{n} \cdot \overrightarrow{OP} = (1, 1, 1) \cdot (1, 0, 0) = 1$ and the plane is

$$x + y + z = 1$$

• Given a function $f(x, y)$ and $P = (a, b)$ in the domain of f, the **tangent plane to the graph $z = f(x, y)$ at $(a, b, f(a, b))$** is the plane defined by the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

• The **linear approximation of $f(x, y)$ at $P = (a, b)$** is the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

For (x, y) near (a, b), $f(x, y) \approx L(x, y)$.

• The **directional derivative of $f(x, y)$ at $P = (a, b)$ in the direction $\mathbf{u}$ (unit vector)** is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(P + h\mathbf{u}) - f(P)}{h} = \nabla f(P) \cdot \mathbf{u}$$

Here $\nabla f(P) = (f_x(P), f_y(P))$ is the **gradient of** $f(x, y)$ **at** P.

• $D_{\mathbf{u}} f(P) > 0 \implies f$ is increasing in the direction $\mathbf{u}$ at P.

• $D_{\mathbf{u}} f(P) < 0 \implies f$ is decreasing in the direction $\mathbf{u}$ at P.

• $D_{\mathbf{u}} f(P) = 0 \implies$ there is no change in f in the direction $\mathbf{u}$ at P.

• $\nabla f(P)$ points in the direction of greatest increase of f near P; $\nabla f(P)$ is perpendicular to the level curve of f through P.