This is the eighth homework assignment for Math 122 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on Wednesday, November 7th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. *Please do Exercises # 3, 5, 18 from Section 13.1 of the textbook.*

Exercise 2. *Please do Exercises # 1-16, 25, 27, 31 from Section 13.2 of the textbook.*

Exercise 3. *Please do Exercises # 1, 3, 7, 9, 11, 17, 21, 23 from Section 13.3 of the textbook.*

Part 2: Turn in on November, 7th

Problem 1 (Limits of functions of two variables)**.** *We recall the definition of a two-variable limit.*

Definition A. *Let $f = f(x, y)$ be a real-valued function of two variables, i.e., $f : \mathcal{D} \to \mathbb{R}$ where f's domain $\mathcal{D}$ is a subset of $\mathbb{R}^2$. Given a point (x_0, y_0) in the domain of f, we say that the two-variable limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) exists and is the number L if $f(x, y)$ can be made as close to L as desired by taking (x, y) as close to (x_0, y_0) as needed. In this case, we write*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

The big (new) idea of a two-variable limit is that $f(x, y)$ must get close to a single number L when looking at all points (x, y) near (x_0, y_0). As we're in two dimensions, there are a lot of directions in which to approach (x_0, y_0) and, for the limit to exist, it is necessary that we get this same value L by studying $f(x, y)$ along any curve $\mathcal{C}$ going through (x_0, y_0). In the language of parameterizations, we state this as a proposition.

Proposition B. *Suppose that the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) exists and is L, i.e.,*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

Then, given any (continuous) parameterization $\mathbf{r}(t) = (x(t), y(t))$ (which parameterizes a curve $\mathcal{C}$ in f's domain) such that

$$\lim_{t\to 0} \mathbf{r}(t) = (x_0, y_0),$$

we must have

$$\lim_{t\to 0} f(\mathbf{r}(t)) = L.$$

The above result is often used to show when a two-dimensional limit does not exist. In this direction, we can study the following equivalent statement (which is the contrapositive of the statement above):

Proposition C. *Let $\mathbf{r}(t)$ and $\mathbf{p}(t)$ be two continuous parameterizations (of two distinct curves) for which*

$$\lim_{t\to 0} \mathbf{r}(t) = \lim_{t\to 0} \mathbf{p}(t) = (x_0, y_0).$$

If

$$\lim_{t\to 0} f(\mathbf{r}(t)) \neq \lim_{t\to 0} f(\mathbf{p}(t)),$$

(which would happen if either of these one-variable limits did not exist) then the two-variable limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) does not exist.

For example, consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

defined for all $(x,y) \in \mathbb{R}^2$*. Let's investigate the existence of the limit of* f *as* (x,y) *approaches* $(0,0)$*. First, we study what happens as* (x,y) *approaches* $(0,0)$ *along the* x*-axis. To this end, we consider the parameterization*

$$\mathbf{r}(t) = (t,0)$$

defined for $t \in \mathbb{R}$*. This curve is obviously continuous and has*

$$\lim_{t\to 0} \mathbf{r}(t) = (0,0).$$

Observe that

$$\lim_{t\to 0} f(\mathbf{r}(t)) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0} \frac{t\cdot 0}{t^2+0^2} = \lim_{t\to 0} 0 = 0.$$

Let's now approach $(0,0)$ *along the line* $y=x$*. This is given the looking at the parameterization* $\mathbf{p}(t) = (t,t)$ *which is continuous and has the property that*

$$\lim_{t\to 0} \mathbf{p}(t) = (0,0).$$

Here we have

$$\lim_{t\to 0} f(\mathbf{p}(t)) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t^2}{t^2+t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since $0 \neq 1/2$*, we conclude that the two-variable limit of* f *as* (x,y) *approaches* $(0,0)$ *does not exist in view of the proposition.*

One may ask, what's so important about looking at arbitrary continuous curves $\mathcal{C}$*? In the example above, we simply evaluated* $f(x,y)$ *along two straight lines and found the the limit was different along two such lines. We can further ask: If the limit of* $f(x,y)$ *is the same along all lines through* $(0,0)$*, does the two-variable limit exist? To settle this question, we consider the function*

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

defined for $(x,y) \in \mathbb{R}^2$*. Please do the following:*

1. *Given any constants* a *and* b*, define* $\mathbf{r}(t) = (at, bt)$ *for* $t \in \mathbb{R}$*. For each fixed* a *and* b*, describe the curve parameterized by* $\mathbf{r}$*.*
2. *Evaluate*
$$\lim_{t\to 0} f(\mathbf{r}(t)).$$
Given your result, which should be independent of a *and* b*, do you think the two-variable limit exists?*
3. *Consider now the parameterization* $\mathbf{p}(t) = (t,t^2)$ *defined for* $t \in \mathbb{R}$*. What curve does this parameterize?*
4. *Compute*
$$\lim_{t\to 0} f(\mathbf{p}(t)).$$
5. *Comparing your results above, what can you conclude about the two-variable limit?*

Problem 2. *Let* $b, c \in \mathbb{R}$*. Consider the quadratic equation*

$$x^2 + bx + c = 0 \tag{$*$}$$

Suppose that this equation has two distinct roots $r_1 < r_2$*.*

1. *Determine a function $f(b, c)$ so that $r_2 = f(b, c)$. Indicate the domain of $f(b, c)$ in the (b, c)-plane.*
2. *Compute the linear approximation $L(b, c)$ of $f(b, c)$ at $(-3, -10)$.*
3. *Explain why the quadratic equation*
$$x^2 - 2.98x - 10.01 = 0$$
has two distinct solutions $s_1 < s_2$. Using linear approximation, estimate s_2. How does your answer compare with the actual value of s_2? You should use a calculator to determine the actual value of s_2.
4. *Use linear approximation to estimate the larger roots of the equations*
$$x^2 - 2.95x - 10 = 0, \qquad x^2 - 3x - 9.95 = 0, \qquad x^2 - 3.2x - 9 = 0$$
How do your estimates compare to the actual values of the larger roots of the above equations?
5. *For the general quadratic equation $(*)$, does a small change in b or a small change in c produce a greater change in the largest root r_2? Justify your answer.*

Problem 3. *In this problem, you will consider an example of a function $g(x, y, z)$ with three input variables - so, the domain of this function is $\mathbb{R}^3$ instead of $\mathbb{R}^2$.*

When patients are screened for osteoporosis, doctors will often do a bone density test. As bones are three-dimensional objects, we need three coordinates to describe a point inside of a bone.

Suppose the bone density (in g/cm^3) at point (x, y, z) is given by the function

$$g(x, y, z) = (x - z)^2 + (y - z)^2 + 0.5.$$

1. *A set of points within the bone which have the same density is called an isosurface or level surface of this function; these are obtained by setting the output value $g(x, y, z) = c$ for some constant density c and taking all points in $\mathbb{R}^3$ which satisfy this equation. In Geogebra3D, plot three different isosurfaces simultaneously, at densities 1, 1.5, and 2 g/cm^3 respectively. Describe these isosurfaces.*
2. *Compute the partial derivative functions $g_x(x, y, z), g_y(x, y, z)$ and $g_z(x, y, z)$.*
3. *Compute the values of each partial derivative at the point $(0, 2, 1)$, and explain the meaning of each quantity in terms of the bone density.*
4. *One common application of derivatives is computing linear approximations to functions. For a single-variable function $f(x)$, the linear approximation at $x = a$ is given by the tangent line:*
$$L(x) = f(a) + f'(a)(x - a).$$
For a two-variable function $f(x, y)$, the linear approximation at $(x, y) = (a, b)$ is given by the tangent plane:
$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$
Extend this idea to a three-variable function $f(x, y, z)$, and find the general formula for the linear approximation $L(x, y, z)$ to $f(x, y, z)$ at point (a, b, c), assuming all three partial derviatives exist.
5. *Using the previous part, give the linear approximation to the bone density function $g(x, y, z)$ at the point $(0, 2, 1)$, and use it to estimate a value for the bone density at point $(0.1, 2.2, 0.9)$. How close is this approximation to the true value given by $g(x, y, z)$?*