This is the seventh homework assignment for Math 122 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on Wednesday, October 31st at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. *Please do Exercises # 13, 43, 44, 53, 55, 60, 61 from Section 12.7 of the textbook. Additionally, please read Example #4 on p. 670.*

Exercise 2. *Please do Exercises #1-7 (odd) 9, 13, 15, 27, 45 from Section 12.9 of the textbook.*

Part 2: Turn in on Wednesday, October 31st

Problem 1. *Points A and B in* $\mathbb{R}^3$ *have Cartesian coordinates* $(1,3,5)$ *and* $(2,2,7)$ *respectively. Find the coordinates of a point C that lies on the line joining A and B, at a distance of 4 units from B.*

Problem 2. *Throughout this problem,* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{R}^3$ *and suppose, additionally, that* $\mathbf{v}$ *is non-zero. The vector projection of* $\mathbf{u}$ *onto* $\mathbf{v}$ *is the vector*

$$Proj_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

Geometrically, $Proj_{\mathbf{v}}\mathbf{u} = (|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|}$*, where* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*:*

We shall denote by $\mathbf{w} = Proj_{\mathbf{v}}\mathbf{u}$ *the vector projection of* $\mathbf{u}$ *onto* $\mathbf{v}$*. Please do the following:*

1. *Show that*
$$|\mathbf{w}| \leq |\mathbf{u}|.$$
Draw a diagram of this property. When is this inequality an equality?

2. *Using properties of the dot product, show that the* $\mathbf{w}$ *is independent of the length of* $\mathbf{v}$ *so as long as* $\mathbf{v}$ *is non-zero. Specifically, show that, for any* $c \neq 0$*, the projection of* $\mathbf{u}$ *on* $c\mathbf{v}$ *is equal to* $\mathbf{w}$*, i.e., show*
$$Proj_{c\mathbf{v}}\mathbf{u} = Proj_{\mathbf{v}}\mathbf{u} = \mathbf{w}.$$

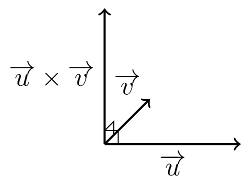
3. *Show that the vector* $\mathbf{w}$ *is a scalar multiple of* $\mathbf{v}$ *(and so is parallel to* $\mathbf{v}$*). Draw a diagram illustrating this.*

4. *Using the dot product, show that the vector* $\mathbf{u} - \mathbf{w}$ *is perpendicular to* $\mathbf{v}$*. Draw a diagram illustrating this property.*

5. *Show that the vector* $\mathbf{u}$ *can be written as a sum of two vectors, one of which is parallel to* $\mathbf{v}$ *and one of which is orthogonal to* $\mathbf{v}$*. Hint: Use your results from the previous parts. Draw a diagram illustrating this property.*

Problem 3. *In this problem you will derive the algebraic formula for the cross product* $\mathbf{u}\times\mathbf{v}$*, where* $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$*. Let* $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$*. The* **cross product of** $\mathbf{u},\mathbf{v}$ *is the vector* $\mathbf{u}\times\mathbf{v}\in\mathbb{R}^3$ *defined as follows:*

(A) (direction) the triple $(\mathbf{u}, \mathbf{v}, \mathbf{u}\times\mathbf{v})$ *follows the* **right-hand rule**: *the vectors* $\mathbf{u}, \mathbf{v}$ *are both perpendicular to* $\mathbf{u}\times\mathbf{v}$ *and are arranged according the the following figure:*



Note: we are not assuming that $\mathbf{u}$ *and* $\mathbf{v}$ *are necessarily perpendicular.*

(B) (magnitude) Let θ *be the angle between* $\mathbf{u}, \mathbf{v}$. *Then, we declare that*

$$|\mathbf{u} \times \mathbf{v}| = \text{ area of parallelogram spanned by } \mathbf{u},\ \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin\theta.$$

An immediate consequence of (A) is the anticommutative property *of the cross product:*

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}, \quad \text{for any } \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \tag{ACP}$$

(You should make sure you can see why this is true.)

We also have the following linearity property *of the cross product:*

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}, \quad \text{for any } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \tag{L1}$$

$$\mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v}), \quad \text{for any } \mathbf{u}, \mathbf{v} \in \mathbb{R}^3,\ c \in \mathbb{R} \tag{L2}$$

For the remainder of this problem we use the following notation for the standard basis vectors:

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

1. *Using a diagram, explain why*

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \tag{$*$}$$

2. *Using (ACP), show that* $\mathbf{u} \times \mathbf{u} = \mathbf{0}$, *for any* $\mathbf{u} \in \mathbb{R}^3$.[1]

3. *Show that*

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}, \quad \text{for any } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \tag{L1'}$$

$$(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}), \quad \text{for any } \mathbf{u}, \mathbf{v} \in \mathbb{R}^3,\ c \in \mathbb{R} \tag{L2'}$$

   You have just shown that the cross product is bilinear. *Hint:* use *(L1), (L2) and (ACP).*

4. *Let* $\mathbf{u} = (a_1, a_2, a_3), \mathbf{v} = (b_1, b_2, b_3) \in \mathbb{R}^3$.

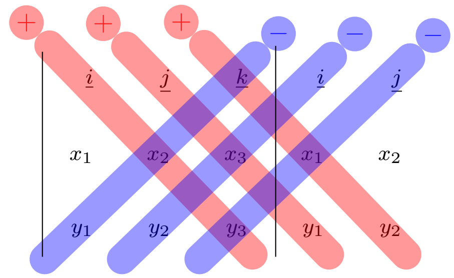
   Using the bilinear properties of the cross product (i.e. properties (L1), (L1'), (L2), (L2')), the anticommutative property (ACP) and $(*)$, *show that*

$$\mathbf{u} \times \mathbf{v} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

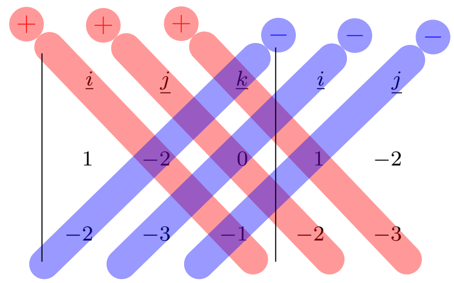
   Hint: write $(a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $(b_1, b_2, b_3) = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

A useful way to visualise how to compute the cross product without remembering the nasty formula is as follows:

[1] This property can also be deduced directly from (B): the parallelogram spanned by $\mathbf{u}$ and itself has zero area.



We multiply across diagonals: for the red diagonals we add the terms, for the blue diagonals we subtract terms. For example, we compute $(1,-2,0)\times(-2,-3,-1)$*:*



$$\begin{aligned}&(1,-2,0)\times(-2,-3,-1)\\ &= (-2)(-1)\underline{i} + 0.(-2)\underline{j} + 1.(-3)\underline{k} - (-2)(-2)\underline{k} - 0.(-3)\underline{i} - 1.(-1)\underline{j}\\ &= 2\underline{i} + \underline{j} - 7\underline{k}\\ &= (2,1,-7)\end{aligned}$$

Problem 4. *In this problem, you will further analyze some properties of the dot product and the cross product.*

a) *Let* $\vec{u}$ *and* $\vec{v}$ *be nonzero vectors in* $\mathbb{R}^3$*. Using the geometric interpretation of the cross product, explain why the following facts will always hold:*

$$(\vec{u}\times\vec{v})\cdot\vec{u} = 0 \qquad\qquad (\vec{u}\times\vec{v})\cdot\vec{v} = 0$$

Then, verify one of these facts algebraically using the formula for the cross product. (The verification for the other fact would look very similar, so you need not do both.)

b) *Consider the vector* $\vec{i} = (1,0,0)$*. For each of the properties below, describe the collection of all* **unit vectors** $\vec{v}$ *which satisfy the property.*

- *The dot product* $\vec{v}\cdot\vec{i}$ *is maximized.*
- *The dot product* $\vec{v}\cdot\vec{i}$ *is zero.*
- *The dot product* $\vec{v}\cdot\vec{i}$ *is minimized.*
- *The cross product* $\vec{v}\times\vec{i}$ *is the zero vector.*
- *The cross product* $\vec{v}\times\vec{i}$ *has the largest possible magnitude.*