This is the sixth homework assignment for Math 122 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on Wednesday, October 24th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. *Please do Exercise #23 from Section 12.5 of the textbook.*

Exercise 2. *Please do Exercises #12, 25, 29, 31 from Section 12.7 of the textbook.*

Exercise 3. *Please do Exercises #7, 11, 15, 17, 29, 31 from Section 13.1 of the textbook.*

Part 2: Turn in on Wednesday, October 24th

In what follows we will generally denote vectors in bold letters, i.e., $\mathbf{v}$, $\mathbf{u}$, $\mathbf{r}$, etc., as opposed to having arrows as you might have seen in lecture (e.g., $\vec{u}$).

Problem 1 (Vector-valued functions, parameterizations and minimal distances)**.** *Though this problem seems long, the text is mostly discussion and is dedicated to walking you through some computations (which you are asked to then repeat/generalize). So please don't be discouraged! This problem is straightforward; all you have to do is the numbered items.*

A vector-valued function is a function $\mathbf{r}$ *mapping from the real line* $\mathbb{R}$ *into* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. These functions are often given by*

$$\mathbf{r}(t) = (x(t), y(t))$$

or

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

defined for real numbers t*, i.e., for* $t \in \mathbb{R}$*. Here* $x(t), y(t)$ *and* $z(t)$ *are simply real-valued (scalar-valued) functions of* t*. You should think about these vector-valued functions as specifying a position in space for each time* t*. In fact, vector-valued functions are some of the main objects studied in classical dynamics (e.g., celestial mechanics) as they describe the motion of an object in space as a function of time. Each such vector-valued function* $\mathbf{r}$ *parameterizes (traces out) a curve* $\mathcal{C}$ *defined by*

$$\mathcal{C} = \{\mathbf{r}(t) : t \in \mathbb{R}\}.$$

We shall say that $\mathbf{r}$ *is a parameterization of* $\mathcal{C}$*. In this course,* $\mathcal{C}$ *will generally be a one-dimensional*[1] *subset of* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. Let's consider, for example, the function* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ *defined by*

$$\mathbf{r}(t) = (\cos(t), \sin(t))$$

for $t \in \mathbb{R}$*. To get a picture of what curve this parameterizes in the xy-plane, observe that*

$$(x(t))^2 + (y(t))^2 = \cos^2(t) + \sin^2(t) = 1$$

for any t*. This is to say that, for all* t*,* $\mathbf{r}(t)$ *lives on the unit circle, i.e., the curve*

$$\mathcal{C} = \{\mathbf{r}(t) : t \in \mathbb{R}\}$$

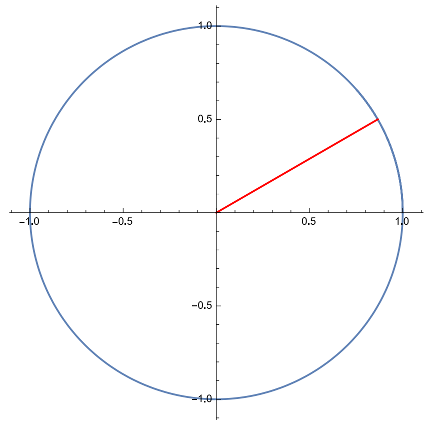


Figure 1: The curve parameterized by $\mathbf{r}(t) = (\cos(t), \sin(t))$.

parameterized by $\mathbf{r}$ *is the unit circle. More is true. In fact, if you plot this curve by letting* t *increase and plot* $\mathbf{r}(t)$ *in* $\mathbb{R}^2$*, you'll see that the unit circle is "traced out" by moving the points at* $\mathbf{r}(t)$ *counterclockwise around the circle:* t *is the angle subtended by the red line and the* x*-axis (in radians). Figure 1 illustrates this.*

Now it's your turn:

1. *Plot the curve in* $\mathbb{R}^3$ *parameterized by*
$$\mathbf{r}(t) = (\cos(t), \sin(t), t)$$
for $t \in \mathbb{R}$*. Hint: If you "projected/collapsed" this curve into the xy plane, you would get a circle because the first two components are exactly what we saw in the example above. So, if you were to walk way up on the* z*-axis and look down, you would essentially see the unit circle. To plot the full curve, think now about what's happening in the* z*-direction as* t *increases.*

We now focus on a special type of vector-valued functions, those which parameterize lines in $\mathbb{R}^2$ *and* $\mathbb{R}^3$*. Let's first focus on the* 2*-dimensional case. Given constants* m *and* b*, consider*

$$\mathbf{r}(t) = (t, mt + b)$$

defined for $t \in \mathbb{R}$*. We note here that* $x = t$ *and* $y = mt + b$*. By making a replacement of* t *for* x *into* y *we obtain*

$$y = mx + b$$

which is, of course, the equation of a line. There is another, perhaps more geometric, way to plot a line in $\mathbb{R}^2$*. If we begin with a point (or vector)* $\mathbf{r}_0 = (x_0, y_0)$ *and a direction vector* $\mathbf{v} = (p, q)$ *we can consider*

$$\mathbf{r}(t) = t\mathbf{v} + \mathbf{r}_0 = t(p, q) + (x_0, y_0) = (pt + x_0, qt + y_0)$$

defined for $t \in \mathbb{R}$*. Note that, when* $t = 0$*,* $\mathbf{r}(0) = 0 \cdot \mathbf{v} + \mathbf{r}_0 = \mathbf{r}_0$*, so the curve parameterized by* $\mathbf{r}$ *contains the point* $\mathbf{r}_0$*. It is not too terribly difficult to see that this parameterizes the line through* $\mathbf{r}_0$ *in the direction of* $\mathbf{v}$*.*

2. *Using the construction above, find the parameterization of the line through* $(0, 1)$ *in the direction* $(1, 1)$*. Draw this line in* $\mathbb{R}^3$ *and indicate where the point* $\mathbf{r}(0)$ *and* $\mathbf{r}(1)$ *are on this line.*

3. *Given two points* (a, b) *and* (c, d) *in* $\mathbb{R}^2$*, find a parameterization* $\mathbf{r}(t)$ *of the line through through the points* $P = (a, b)$ *and* $Q = (c, d)$*. Hint: The direction vector from* $P = (a, b)$ *to* $Q = (c, d)$ *is* $\mathbf{v} = (c - a, d - b)$*.*

[1] It's not always the case the these curves are one-dimensional, which is a rather curious phenomenon. If this is of interest to you, look up space-filling curves or fractal geometry.

Now that we can describe lines parametrically in $\mathbb{R}^2$, we can, in fact, ask some interesting questions. For example, given the line parameterized by $\mathbf{r}(t) = t(1,1) + (0,-6)$, can we find a point on the line which is closest to the point $(1,1)$? In other words, among all point on the line parameterized by $\mathbf{r}(t)$, can we find the point that minimizes the distance from this line to the point $(1,1)$?

Sure, we can! Given any t, the distance from the point $(1,1)$ to the point (on the line) $\mathbf{r}(t)$ is

$$d(t) = \|\mathbf{r}(t) - (1,1)\| = \|(t, t-6) - (1,1)\| = \|(t-1, t-7)\| = \sqrt{(t-1)^2 + (t-7)^2}.$$

We are interested in where this distance is minimized, i.e., for which t is $d(t)$ the smallest. A moment's thought shows that $d(t)$ is minimized if and only if $(d(t))^2 = d^2(t)$ is minimized, so our goal is to minimize the function

$$d^2(t) = (t-1)^2 + (t-7)^2 = 2t^2 - 16t + 50.$$

This is, of course, a Calculus 1 problem and the minimum is easily found by finding the function's critical point. We have

$$0 = \frac{d}{dt} d^2(t) = \frac{d}{dt}\left(2t^2 - 16t + 50\right) = 4t - 16$$

which yields $t = 4$ as a critical point. It is easy (with the second derivative test) to check that this is a minimum. Consequently, the function $d(t)$ is minimized when $t = 4$ from which we obtain the minimum distance

$$d(4) = \sqrt{(4-1)^2 + (4-7)^2} = \sqrt{9+9} = 3\sqrt{2}.$$

which is attained at the point

$$\mathbf{r}(4) = (4, 4-6) = (4, -2).$$

Now, it's your turn to extend these ideas into $\mathbb{R}^3$.

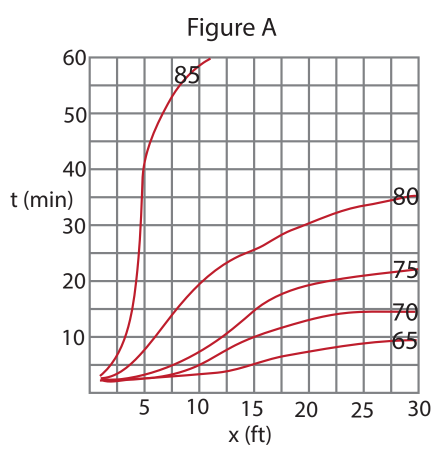
4. *Given a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a direction $\mathbf{v} = (p, q, r)$ find a parameterization of the line through $\mathbf{r}_0$ in the direction $\mathbf{v}$.*

5. *Given points (a, b, c) and (d, e, f), find a parameterization of the line through (a, b, c) and (d, e, f).*

6. *Find the minimum distance from the line through $(-1, 0, 0)$ and $(0, 1, 2)$ to the point $(0, 0, 7)$. Find the point at which this minimum distance is attained.*

Problem 2. *In this problem, you will investigate a specific contour diagram from an unspecified function.*

You are in a room 30 feet long with a heater on the wall at one end. In the morning, the temperature in the room is 65°F. You turn on the heater, which quickly warms up to 85°F. Let $H(x,t)$ be the temperature x feet from the heater t minutes after the heater is turned on. Figure A shows the contour diagram for H.



1. *Using the contour diagram, about how warm is it 15 feet from the heater 20 minutes after it was turned on?*

2. *On a set of coordinate axes, draw and label sketches of the cross-sections $H(x, 10)$ and $H(x, 30)$. Be sure to label your axes! Describe what these two cross-sections represent. How can you explain the difference between these two graphs?*

3. *On another set of coordinate axes, draw and label sketches of the cross-sections $H(10, t)$ and $H(30, t)$. Be sure to label your axes! Describe what these two cross-sections represent. How can you explain the difference between these two graphs?*

4. *Imagine that 10 minutes after the heater is turned on, you and your cat Fermat are sitting 10 feet from the heater. How will H change as time goes on, if you stay in the same place? How do you know?*

5. *Imagine that 20 minutes after the heater has been turned on, Fermat gets up and walks quickly away from the heater. How will H change for Fermat as he walks? How do you know?*

6. *Sketch a new contour diagram for the function $T_{10}(a, b)$, which gives the temperature at point (a, b) in the room 10 minutes after the heater was turned on. Indicate the location of the heater in your picture. How does this new contour diagram differ from the original?*[2]

Problem 3. *In this problem you will determine a function $D(x, y, z)$ that computes the distance from an arbitrary point $(x, y, z) \in \mathbb{R}^3$ to an arbitrary plane $\mathcal{P}$. Here,* distance means shortest distance.

1. *Determine the distance from $P = (u, v, w)$ to each of the coordinate planes (i.e. the xy-, xz- and yz-planes). Your answer should be in terms of u, v, w.*

2. *Consider the plane defined by*

$$2x - 4y + 3z = 2.$$

   Determine the distance from $P = (u, v, w)$ to this plane. Your answer should be in terms of u, v, w.

3. *Consider a general plane*

$$\mathcal{P} : \quad \mathbf{n} \cdot (x, y, z) = d$$

   Here $\mathbf{n} = (a, b, c)$ is a normal vector for the plane. Determine a formula for the function

$$D : \mathbb{R}^3 \to \mathbb{R}\ ,\ (u, v, w) \mapsto D(u, v, w)$$

   where $D(u, v, w) =$ distance from (u, v, w) to $\mathcal{P}$. Your answer should be in terms of $\mathbf{n}$ and $\mathbf{u} = (u, v, w)$.

[2] The coordinates (a, b) here are meant to be coordinates in the x-y plane - you may ignore the fact that the room has height and the fact that heat rises, and assume temporarily that heat is determined entirely by your position in the room.