This is the second homework assignment for Math 122 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on Wednesday, September 19th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. *Please do Exercise 1, Parts a-d, from Section 11.2 of the textbook.*

Exercise 2. *Please do Exercises 7, 13 and 17 from Section 11.2 of the textbook.*

Exercise 3. *Please do Exercises 33 and 45 from Section 11.2 of the textbook.*

Part 2 (Solutions for these problems are due in class on September 19th)

Problem 1. *As we've often seen, determining when/if a sequence converges can be a delicate matter. Many of the results in Section 11.1 aim to help with this task, e.g., Theorem 1 (Algebra with limits), Theorem 2 (The squeeze principle/Theorem) and Theorem 3 (Monotonic Bounded Theorem) in Section 11.1. Another result that's often of great use is the following.*

Theorem A. *Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with $\lim_{n\to\infty} a_n = L$. If f is a continuous function (or is, at least, continuous at L), then the sequence $\{f(a_n)\}_{n=1}^{\infty}$ is convergent and*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

As an example of the utility of this theorem, consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by

$$b_n = \ln\left(1+\frac{1}{n}\right)$$

for $n = 1, 2, \ldots$. At a cursory glance, it's not completely clear what's happening with this sequence. We can, however, easily recognize that $b_n = \ln(a_n)$ where $a_n = 1 + 1/n$. Now, in view of Theorem 1 in Section 11.1, $\{a_n\}_{n=1}^{\infty}$ is convergent with $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1 + 1/n) = 1$. Since $f(x) = \ln(x)$ is a continuous function at $x = 1$, we can conclude that $b_n = f(a_n)$ is itself convergent and use Theorem A to deduce

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln(\lim_{n\to\infty} a_n) = \ln(1) = 0.$$

Now, it's your turn. Make sure to carefully justify the steps involved in your solution.

1. *Use the theorem above to compute*
$$\lim_{n\to\infty} e^{(1/n+\ln(2))}.$$
2. *Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by*
$$a_n = \left(1+\frac{x}{n}\right)^n$$
for $n = 1, 2, \ldots$. Show that $\{a_n\}_{n=1}^{\infty}$ is convergent and find its limit. Hint: You can show directly (using properties of the natural logarithm and L'Hôpital's rule) that $\lim_{n\to\infty} \ln(a_n) = x$. Noting also that $a_n = e^{\ln a_n}$, your result should follow from the theorem above and the observation that the exponential function is continuous.

3. *Given a sequence* $\{a_n\}_{n=1}^{\infty}$ *and a continuous function* f*, is it possible that* $\lim_{n\to\infty} f(a_n)$ *exists even if* $\{a_n\}_{n=1}^{\infty}$ *is not convergent? If so, give an example to demonstrate your claim; if not, explain.*

Problem 2. *In this problem, we investigate the convergence of series via its definition, i.e., in terms of the convergence/divergence of its sequence of partial sums.*

1. *First, consider a series* $\sum_{n=1}^{\infty} a_n$ *whose partial sums* $\{S_n\}_{n=1}^{\infty}$ *are known to satisfy*

$$0 \le S_n \le 100 \tag{1}$$

for $n = 1, 2, \ldots$.

(a) *Give an example of a sequence* $\{a_k\}_{k=1}^{\infty}$ *whose partial sums satisfy the above inequality* (1) *but for which the series* $\sum_{k=1}^{\infty} a_k$ *is divergent.*

(b) *Suppose that, in addition to the inequality* (1)*, it is known that if* $a_k \ge 0$ *for all* $k = 1, 2, 3, \ldots$. *Show that the series* $\sum_{k=1}^{\infty} a_k$ *must converge.*

2. *We now turn to a specific example. Suppose that the partial sums of a series* $\sum_{k=1}^{\infty} b_k$ *are given by*

$$S_n = \ln\left(\frac{2n}{n+1}\right)$$

for $n = 1, 2, \ldots$.

(a) *Evaluate* $\lim_{n\to\infty} S_n$.

(b) *Does the series* $\sum_{k=1}^{\infty} b_k$ *converge? Explain your reasoning (no more than one sentence needed).*

(c) *Find a formula for the terms* b_k *of the series* $\sum_{k=1}^{\infty} b_k$ *and simplify as much as possible.*

(d) *Using your formula in (c), show directly that* $\lim_{k\to\infty} b_k = 0$.

Problem 3. 1. *Consider the series*

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$$

(a) *Find constants* c, d *so that*

$$\frac{1}{(2k-1)(2k+1)} = \frac{c}{2k-1} + \frac{d}{2k+1}$$

(b) *Using a telescoping argument, show that the series is convergent and determine its limit.*

2. *In this problem you will investigate the relationship between repeating decimals and fractions.*

(a) *Let* $x = 0.333\ldots$.

i. *Express* x *as a geometric series.*

ii. *Determine the limit of this geometric series.*

(b) *Let* $y = 0.ababab\ldots$*, where* $0 \le a, b \le 9$ *are integers.*

i. *Express* y *as a geometric series.*

ii. *Show that* $y = \frac{10a+b}{99}$.

iii. *Use the previous problem to determine the decimal expansion of* $\frac{8}{33}$.

Problem 4. *In this problem, you will give an alternative explanation for why the harmonic series diverges.*

Let

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

be the n*th partial sum of the harmonic series. Also, let*

$$I_n = \int_1^{n+1} \frac{1}{x}\, dx$$

be the integral of $\frac{1}{x}$ *from* 1 *to* $n+1$.

a) *Show that* S_n *actually computes a left-hand/upper Riemann sum approximation to* I_n *with exactly* n *equal-sized subdivisions.*

b) *Use part (a) to compare* S_n *and* I_n *(a picture may be helpful). Use the relationship between these two quantities to show that the harmonic series* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges.*

   Hint: You may find it helpful to use a result from your previous homework (Problem 1 on Homework 1).