This is the first homework assignment for Math 122 and it is broken into two parts. The first part of the homework consists of exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on Wednesday, September 12th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. *Do Exercises* $1-15$ *(odd) in Section 11.1 in the textbook.*

Exercise 2. *Do Exercise 18 in Section 11.1 in the textbook.*

Part 2 (Solutions for these problems are due in class on September 12th)

Problem 1 (Improper Integrals)**.** *At first, when the Riemann integral is introduced in Calculus 1, one only considers integrating a function* f *on a closed and bounded interval* $[a,b]$*. In this case, the Riemann integral of a function* f *on* $[a,b]$ *is*

$$\int_a^b f(t)\,dt$$

and is defined as the limit of Riemann sums. So, at least at first, Riemann integration doesn't tell you how to consider integrals like

$$\int_0^\infty f(t)\,dt \qquad \text{or} \qquad \int_{-\infty}^\infty f(t)\,dt.$$

Let's remedy this. Let $f:[a,\infty)\to\mathbb{R}$ *be a continuous function (or piecewise continuous function), i.e.,* f *is a function from the interval* $[a,\infty)$ *into the set of real numbers* $\mathbb{R}=(-\infty,\infty)$ *which is continuous at every point. Given any fixed* $x>a$*, it makes sense to consider the integral*

$$\int_a^x f(t)\,dt$$

because f *is continuous on* $[a,x]$*. More precisely, the continuity of* f *on* $[a,x]$ *guarantees that* f *is integrable (or Riemann-integrable) on* $[a,x]$*. We note that, to each such* $x>a$*,*

$$I(x)=\int_a^x f(t)\,dt$$

is a real number and hence I *is a real-valued function on* (a,∞)*. If the limit*

$$\lim_{x\to\infty} I(x)=\lim_{x\to\infty}\int_a^x f(t)\,dt$$

exists, we say that the improper Riemann integral of f *on* $[a,\infty)$ *converges and write*

$$\int_a^\infty f(t)\,dt=\lim_{x\to\infty} I(x)=\lim_{x\to\infty}\int_a^x f(t)\,dt.$$

The number $\int_a^\infty f(t)\,dt$ *is said to be the improper Riemann integral of* f *on* $[a,\infty)$*.*

Let's work out an example. It will be instructive to follow this example carefully, as you'll be asked to do three similar ones (below). Consider the function $f:[1,\infty)\to\mathbb{R}$ *defined by*

$$f(t)=\frac{1}{t^2}$$

for $t \geq 1 = a$. *Our goal is to understand whether or not the improper Riemann integral*

$$\int_1^\infty f(t)\,dt = \int_1^\infty \frac{1}{t^2}\,dt$$

converges. Of course, the function f *is continuous on the interval* $[1, \infty)$ *and so it is ripe for this type of investigation. To sort things out, for any number* $x > 1$, *we compute*

$$I(x) = \int_1^x \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_1^x = 1 - \frac{1}{x}.$$

In this case,

$$\lim_{x\to\infty} I(x) = \lim_{x\to\infty}\left(1 - \frac{1}{x}\right) = 1 - 0 = 1$$

and so, in view of the definition of the improper Riemann integral, we can conclude that the improper integral in question converges and

$$\int_1^\infty \frac{1}{t^2}\,dt = \lim_{x\to\infty} I(x) = 1.$$

Now it's your turn:

1. *By the definition above, determine whether or not the following improper integrals converge and, if so, give their value.*

   a.
   $$\int_1^\infty \frac{1}{t^{3/2}}\,dt$$

   b.
   $$\int_1^\infty \frac{1}{t}\,dt$$

   c.
   $$\int_0^\infty e^{-t}\,dt$$

   d.
   $$\int_0^\infty g(t)\,dt \qquad \text{if} \qquad g(t) = \begin{cases} t^2 & 0 \leq t \leq 4 \\ 0 & 4 < t < \infty \end{cases}$$

2. *Use properties of limits (from Calc 1) to prove/justify the following fact:*

   Fact A. *Let* h *and* k *be continuous real-valued functions on the interval* $[a, \infty)$. *If the improper integrals*

   $$\int_a^\infty h(t)\,dt \qquad \text{and} \qquad \int_a^\infty k(t)\,dt$$

   converge, then the associated improper Riemann integral of $h + k$ *on* $[a, \infty)$ *converges and*

   $$\int_a^\infty (h(t) + k(t))\,dt = \int_a^\infty h(t)\,dt + \int_a^\infty k(t)\,dt.$$

3. *Use the fundamental theorem of calculus to prove/justify the following fact.*

   Fact B. *Let* f *be a continuous function on* $[a, \infty)$ *with antiderivative* F, *i.e.,* $F'(x) = f(x)$ *for all* $a < x < \infty$. *If the limit*

   $$\lim_{x\to\infty} F(x)$$

   exists, then the improper Riemann integral of f *on* $[a, \infty)$ *converges and*

   $$\int_a^\infty f(t)\,dt = \lim_{x\to\infty} F(x) - F(a).$$

Problem 2 (Some good ways to find limits)**.** *In this problem, we will study a couple of interesting ways to find limits.*

1. *Consider the sequence* $\{a_n\}$ *defined by*

$$a_n = \sum_{k=1}^{n} \frac{k}{n^2}$$

for $n = 1, 2, \ldots.$

   (a) *Find the values of* a_5 *and* a_{10} *of this sequence.*

   (b) *Thinking back to Calculus 1 and, in particular, Riemann integration and Riemann sums, explain why* $\{a_n\}$ *must be a convergent sequence and*[1]

$$\lim_{n\to\infty} a_n = \int_0^1 x\,dx = \frac{1}{2}.$$

2. *Consider the sequence* $\{b_n\}$ *defined by*

$$b_n = \frac{1}{n^2} \sin\left(n\frac{\pi}{12}\right)$$

for $n = 1, 2, \ldots.$

   (a) *Find the values of* b_3 *and* b_6 *of this sequence.*

   (b) *Does the sequence* $\{b_n\}$ *converge? If your answer is "no", explain your reasoning. If your answer is "yes", find the value of the limit. Hint: You might want to play around with the inequality* $-1 \leq \sin(\theta) \leq 1$ *for all* θ*.*

Problem 3. *In this problem, we will make use of the* $\epsilon - N$ *definition of convergence of a sequence:*

Definition C. *Let* $\{a_k\}$ *be a sequence of real numbers. We say that* $\{a_k\}$ *converges to a real number* L *if for any* $\epsilon > 0$*, there is some natural number* N_ϵ *so that*

$$\text{for all } k \geq N_\epsilon, \text{ we have } |a_k - L| < \epsilon.$$

In this case, we say L *is the limit of the sequence, and write* $\lim_{k\to\infty} a_k = L$*. If no such number* L *exists, then we say the sequence diverges.*

More informally, this means that the sequence converges to L *if, no matter how close we need to be to* L *(*ϵ *measures this closeness), there is some place in the sequence (*N_ϵ *gives us this place) beyond which we are always and forever at least that close to* L*.*

In this exercise, you will be given some values for ϵ*, our closeness value, and you will need to find a place in the sequence (*N_ϵ*) which guarantees that closeness.*

Consider the sequence $\{\frac{1}{2^k}\}_{k=0}^{\infty}$*.*

1. *To what limit* L *does this sequence converge?*

2. *Find* N *so that if* $k \geq N$*, then* a_k *is within* $\frac{1}{8}$ *of* L*. That is, find* $N_{\frac{1}{8}}$ *Be sure to explain not only why* a_N *is within* $\frac{1}{8}$ *of* L*, but how you know that all the terms appearing later in the sequence will also be within* $\frac{1}{8}$ *of* L*.*

3. *Find* $N_{\frac{1}{100}}$ *(and again, explain).*

4. *Give the best general formula you can find for* N_ϵ*, and explain your reasoning.*

Problem 4. *Recall the following theorem:*

[1] Hint: $k/n^2 = (k/n) \cdot (1/n)$.

Theorem D (Monotonic Bounded Theorem)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a monotonic and bounded sequence of real numbers. Then, $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence.*

The Monotonic Bounded Theorem is really useful because it allows us to show that a sequence is convergent without having to identify the limit (i.e. we don't have to find the real number L given in the definition from the previous exercise).

In this exercise we will investigate the Monotonic Bounded Theorem and formulate a generalisation. It's time to flex our mathematical muscles:

1. *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be two sequences, both of which are monotonic and bounded.*
   (a) *Explain why the sequence $\{c_n\}_{n=1}^{\infty}$, where, for each $n = 1, 2, 3, \ldots$, $c_n = a_n + b_n$, is convergent.*
   (b) *It is not always the case that the sequence $\{c_n\}$ given in the previous problem is monotonic; $\{c_n\}$ is always bounded, however. Give examples of monotonic and bounded sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ so that $\{c_n\}_{n=1}^{\infty}$, where, for each $n = 1, 2, 3, \ldots$, $c_n = a_n + b_n$, is* **not** *monotonic*

2. *Consider the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$, where*

$$x_n = \begin{cases} \sin(1/n), & 1 \leq n \leq 4, \\ -\frac{1}{n}, & n \geq 5 \end{cases}$$

   (a) *Graph the first eight terms of the sequence $\{x_n\}$.*
   (b) *Is the sequence $\{x_n\}$ monotonic? bounded? convergent? Justify your answer.*
   (c) *Suppose we changed the definition of $\{x_n\}$ to the following:*

$$x_n = \begin{cases} \sin(1/n), & 1 \leq n \leq 100, \\ -\frac{1}{n}, & n \geq 101 \end{cases}$$

   Is this redefined sequence monotonic? bounded? convergent? Justify your answer. (Hint: what would the graph of the the redefined sequence look like?)

   (d) *Explain carefully why the sequence of real numbers $\{y_n\}_{n=1}^{\infty}$ is convergent, where*

$$y_n = \begin{cases} \frac{\sin(n)}{n^2}, & 1 \leq n < 10^{10^{10}}, \\ -\frac{1}{n}, & n \geq 10^{10^{10}} \end{cases}$$

   (Hint: consider carefully your solutions to (a)-(d))

   (e) **True/False:** *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If there is some natural number $K \geq 1$ so that the sequence $\{x_n\}_{n=K}^{\infty}$ is monotonic and bounded then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent.*
   Justify your answer using, at most, two (coherent) sentences and/or an appropriate diagram.