This is the eleventh homework assignment for Math 122 and it is broken into two parts. The first part of the homework consists of textbook exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on Wednesday, December 5th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. *Please do Exercises # 1 (only a), 3 (only a), 25 and 29 from section 14.1*

Exercise 2. *Please do Exercises #1-13 odd, 15, 17, 23, from section 14.2*

Exercise 3. *Please do Exercises #9-19 odd, 23, 29, 31 from section 14.3*

Part 2 (Turn this in!)

Problem 1. *In this problem you will think about changing the order of integration.*

1. *Consider the double integral*

$$I = \int\int_R \frac{e^{x/y}}{y} dA = \int_0^1 \int_x^{\sqrt{x}} \frac{e^{x/y}}{y} dy dx$$

   (a) *Sketch the region R.*

   (b) *What issue arises when you try to compute I using the indicated iterated integral?*

   (c) *By changing the order of integration, determine I.*

2. *Since changing the order of integration corresponds to changing the order of certain limits, it should not be surprising to know that this operation holds for functions that are 'not too badly behaved'.*

   Fubini's Theorem: *if $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$ then*

$$\int\int_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy$$

   In this problem, you will see an instance where changing the order of integration does change the value of a double integral.

   (a) *Show that*

$$\frac{\partial}{\partial x}\left(\frac{-x}{(x+y)^2}\right) = \frac{x-y}{(x+y)^3}, \qquad \frac{\partial}{\partial y}\left(\frac{y}{(x+y)^2}\right) = \frac{x-y}{(x+y)^3}$$

   (b) *By evaluating both double integrals, show that*

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dx dy \neq \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dy dx$$

   (c) *Explain why $f(x,y) = \frac{x-y}{(x+y)^3}$ is not continuous on $[0,1] \times [0,1]$.*

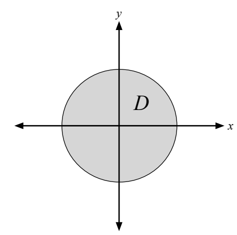
Problem 2. *For each of the following double integrals, sketch the region of integration R. Then evaluate the integral* **by hand**, *explaining your process.*

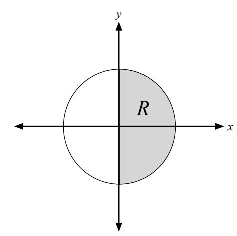
1. $\displaystyle\int_0^1 \int_2^4 \int_{-3}^1 3\,dz\,dx\,dy$

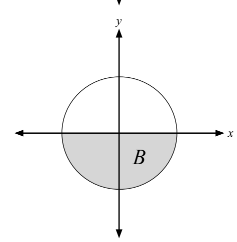
2. $\displaystyle\int_0^2 \int_{x^2}^4 x\cos(y^2)\,dy\,dx$

3. $\displaystyle\int_0^1 \int_0^1 \sin(e^x)\,dx\,dy + \int_1^e \int_{\ln(y)}^1 \sin(e^x)\,dx\,dy$

Problem 3. *Let D be the region inside the unit circle centered at the origin, let R be the right half of D, and let B be the bottom half of D. Decide, without calculating the value of any of the integrals, whether each integral is positive, negative, or zero. Explain your answers.*







1. $\displaystyle\iint_D 1\,dA$

2. $\displaystyle\iint_R 5x\,dA$

3. $\displaystyle\iint_B 5x\,dA$

4. $\displaystyle\iint_D e^x\,dA$

5. $\displaystyle\iint_R x^2y^3\,dA$

Problem 4. *Before publishing his groundbreaking theory of gravitation, Isaac Newton spent a great deal of time worrying*[1]*. Newton's theory of gravity says that the gravitational force on an object of mass m due to an object of mass M is inversely proportional to the square of the distance between those objects. In mathematical form, this is*

$$\mathbf{F} = -G\frac{mM}{r^2}\mathbf{e}_r$$

where $\mathbf{e}_r$ is a unit vector in the direction from M to m, r is the distance between the objects and G is a constant of proportionality, called the gravitational constant. As it turns out, this force (which is a vector) is gotten from the gradient of a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\phi(r) = \phi(x,y,z) = -G\frac{M}{r} = -G\frac{M}{\sqrt{x^2+y^2+z^2}} \qquad (1)$$

where we shall assume that the object of mass M sits at the origin $(0,0,0)$ and the object of mass m sits at the point (x,y,z), a distance $r = \sqrt{x^2+y^2+z^2}$ from $(0,0,0)$.

[1] Classical Dynamics of Particles and Systems. S. Thornton and J. Marion

1. Show that

$$\mathbf{F} = m\nabla\phi.$$

Hint: You should note that

$$\mathbf{e}_r = -\frac{(x,y,z)}{|(x,y,z)|} = -\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

This function ϕ is called the gravitational potential for the mass M. In some sense, it determines the motion of all massive objects in space[2]*. And for this reason, we focus out study on ϕ.*

We can now discuss the source of Newton's great worry. Though Newton's theory worked spectacularly[3]*, it treats massive objects (e.g., our Sun) as points in space without mass or volume. This is, of course, nonsense. To account for realistic massive objects, the gravitational potential must be defined using a triple integral where mass is treated as being spread continuously over the space it occupies. For example, if the object is a sphere or radius 1 centered at the origin $(0,0,0)$, the gravitational potential is*

$$\phi(r) = -G\int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{\rho\lambda^2}{R}\sin(\varphi)\,d\lambda\,d\varphi\,d\theta \tag{2}$$

where $R^2 = \lambda^2 + r^2 - 2r\lambda\cos\varphi$ is found using the law of cosines. In this formula (which might look slightly intimidating, but is pretty simple), ρ is a constant called the mass density and the total mass of the sphere is defined by

$$M = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho\lambda^2\sin(\varphi)\,d\lambda\,d\varphi\,d\theta.$$

2. Perform the integration in the equation above for M thus finding a simple relationship between the total mass M and the mass density ρ.

Looking back to Equation (2), we can invoke Fubini's theorem[4] *and simplify to find*

$$\begin{aligned}\phi(r) &= -G\int_0^1\int_0^{\pi}\int_0^{2\pi} \frac{\rho\lambda^2}{R}\sin(\varphi)\,d\theta\,d\varphi\,d\lambda \\ &= -G\rho\int_0^1\int_0^{\pi}\int_0^{2\pi} \frac{\lambda^2}{R}\sin(\varphi)\,d\theta\,d\varphi\,d\lambda.\end{aligned}$$

Upon noting that the integrand $\lambda^2\sin(\varphi)/R$ is independent of θ, we can iterate the integrals (and carry out the θ integration) to find

$$\phi(r) = -2\pi G\rho\int_0^1 \lambda^2\left(\int_0^{\pi}\frac{1}{R}\sin(\varphi)d\varphi\right)d\lambda \tag{3}$$

where (still) $R^2 = \lambda^2 + r^2 - 2r\lambda\cos(\varphi)$. We isolate the innermost integral

$$\int_0^{\pi}\frac{1}{R}\sin(\varphi)\,d\varphi \tag{4}$$

[2] In fact, ϕ satisfies Laplace's equation, $\Delta\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$ (which you should check if you're sufficiently interested).

[3] It predicts Kepler's Laws, for example, though, as Einstein showed, it's not the final word.

[4] You can take for granted that Fubini's Theorem is valid in this case.

and, by treating λ and r as constant, we make a change of variables from φ to R where the relation is specified by

$$R^2 = \lambda^2 + r^2 - 2r\lambda\cos(\varphi)$$

and so

$$2R\,dR = 2r\lambda\sin(\varphi)\,d\varphi.$$

3. *Perform this change of variables on the integral (4). You should obtain an integral of the form*

$$\int_{R_0}^{R_1} (\lambda \textit{ and } r \textit{ stuff})\,dR$$

   where R_0 and R_1 are also functions of r and λ.

4. *Insert your result from the previous step back into Equation (3) and simplify as much as possible (without, yet, performing the double integral).*

5. *In the case that $r > 1$, i.e., you're viewing the gravitational potential from outside the sphere, the limits of integration R_0 and R_1 should simplify considerably upon noting that $0 \leq \lambda \leq 1 < r$. Under this simplification, compute the double integral and simplify. Your resulting expression $\phi(r)$ should now only be a function of r, G and ρ.*

6. *Finally, using your result from Part 2, put this expression for $\phi(r)$ in terms of r, G and M. Comparing your expression from what we obtained in (1), does the gravitational potential look any different away from a point mass as it does from a sphere? Comment on why Newton can stop worrying.*