This is the tenth homework assignment for Math 122 and it is broken into two parts. The first part of the homework consists of textbook exercises you should do (and I'll expect you to do) but you needn't turn in. As these exercises will not be graded, if you would like help with them or just want to make sure you're doing them correctly, you should (always) feel free to come to office hours (mine or those of the TAs). The second part is the part you are expected to turn in. More precisely, please complete all problems in Part 2, write up clear and thorough solutions for them (consistent with the directions given in the syllabus) and hand them in. Your write-ups are due on Wednesday, November 28th at the beginning of class. As always, please come and see me early if you get stuck on any part of this assignment. I am here to help!

Part 1 (Do not turn in)

Exercise 1. *Please do Exercises #1-7 odd, 9, 14, 23, 37 from section 13.5.*

Exercise 2. *Please do Exercises #3, 5, 7, 13, 15, 17, 23, 27 from section 13.6*

Exercise 3. *Please do Exercises #1, 2, 3, 4, 5, 7, 9, 17, 19 from section 13.7*

Part 2 (Turn this in!)

Problem 1. *It snowed last night, so today all the Colby students head to the local sledding hill, whose surface is given by $f(x, y) = 2y\cos(x)$. The students all stand at the point $(0, 0)$ to start with.*

1. *Plot a picture of the sledding hill, and indicate where the students are standing.*
2. *In which direction should the students point their sleds for the steepest downhill trajectory? How steep will the slope be in this direction? Justify your answer mathematically.*
3. *If the students point their sleds in the direction of $\vec{v} = \vec{i} + \vec{j}$, what will happen? Justify your answer mathematically.*
4. *One student doesn't like sledding, and wants to take a nice easy walk without going uphill or downhill. In which direction could he walk? How long would he be able to continue this way without going up or downhill?*
5. *If you were one of these students, where would you choose to start your sledding adventure? Why?*

Problem 2. *Consider the function f defined by*

$$f(x, y) = \sqrt{x}(\sin(y) + 1)$$

for $x > 0$ and $y \in \mathbb{R}$. In this problem, we shall approximate f using the linear and quadratic approximations at $(1, 0)$.

1. *Compute the local linearization*

$$L(x, y) = \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) + f(x_0, y_0)$$

   for f at $(x_0, y_0) = (1, 0)$.

2. *Compute the quadratic approximation*

$$\begin{aligned} Q(x, y) &= L(x, y) + \frac{1}{2}\begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix}\begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \\ &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &\quad + \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2 \end{aligned}$$

for f at $(x_0, y_0) = (1, 0)$. Simplify as much as possible.

3. *Show that f and the quadratic approximation Q share all of the same first and second order partial derivatives at $(x_0, y_0) = (1, 0)$. Note: This requires you to compare numbers for 6 different partial derivatives, two first-order ones and four second-order ones.*

4. *Let's now use the linear and quadratic approximations to estimate the number*

$$f(0.9, 0.1) = \sqrt{0.9}(\sin(0.1) + 1)$$

   (a) *By hand calculation, compute $L(0.9, 0.1)$.*

   (b) *By hand calculation, compute $Q(0.9, 0.1)$.*

   (c) *Using Wolfram Alpha (or something equivalent), compute the actual number $\sqrt{0.9}(\sin(0.1) + 1)$ accurate to 10 digits. Make a quantitative remark about the approximations with L and Q, i.e., which one is closer to the actual value and by how much?*

Problem 3. *In this problem you will derive an important result in an area of statistics. This is the theory of lines of best fit and plays a prominent role in data analysis, machine learning, quantitative methods in the natural sciences, among many other quantitative fields.*

Background

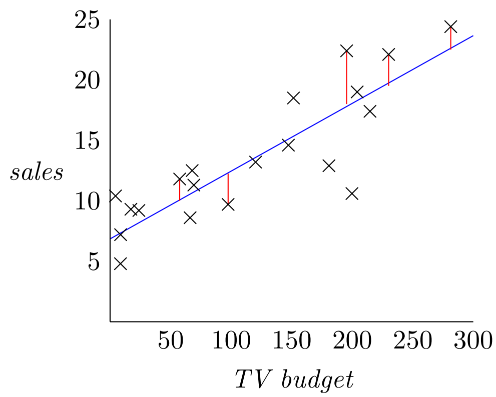
Consider a collection of n items of data of the form

$$(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$$

We obtain a scatterplot by plotting this data in the xy-plane. For example, the following table represents (real) data relating twenty company's total television advertising budgets (in thousands of dollars) to sales of their product (in thousands of units)

TV Budget	Sales
230.1	22.1
44.5	10.4
17.2	9.3
151.5	18.5
180.8	12.9
8.7	7.2
57.5	11.8
120.2	13.2
8.6	4.8
199.8	10.6
66.1	8.6
214.7	17.4
23.8	9.2
97.5	9.7
204.1	19
195.4	22.4
67.8	12.5
281.4	24.4
69.2	11.3
147.3	14.6

Let x_i denote the TV budget of company i (in thousands of dollars) and y_i denote the sales of the product made by company i (in thousands of units). The resulting scatterplot is given below:



Each $\times$ represents a data point. Also represented on the scatterplot is the (least squares) line of best fit: this is the unique line $y = mx + b$ that minimises the average of the sum of the squares of the lengths of the red lines. This quantity is known as the root mean square error (RMSE):

$$RMSE = \sqrt{\frac{(y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \ldots + (y_n - mx_n - b)^2}{n}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - mx_i - b)^2}$$

To determine the (least squares) line of best fit we have to find m, b that minimises the RMSE. To do this, it is sufficient to determine the global minimum of the function

$$f(m, b) = (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \ldots + (y_n - mx_n - b)^2 = \sum_{i=1}^{n}(y_i - mx_i - b)^2$$

Let $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ be a given data set. In particular, all the x_i's and y_i's are numerical values (i.e. constants).

1. *Show that $f(m, b)$ has a unique stationary point (m_0, b_0), where*

$$m_0 = \frac{(\sum_{i=1}^{n} x_i y_i) - n\overline{x}\overline{y}}{(\sum_{i=1}^{n} x_i^2) - n\overline{x}^2}, \qquad b_0 = \overline{y} - m_0\overline{x}$$

   Here

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

   are the mean averages of the x_i's and y_i's.

2. *Show that (m_0, b_0) is a local minimum using the second derivative test. Hint: you will need the following useful formula*

$$(n-1)\sum_{i=1}^{n} x_i^2 - 2\sum_{1 \le i < j \le n} x_i x_j = \sum_{1 \le i < j \le n} (x_i - x_j)^2$$

3. *Explain why this local minimum must be a global/absolute minimum for $f(m, b)$.*

4. *For the TV advertising/sales data given above show that the slope of the (least squares) line of best fit is positive. Does TV advertising work? Justify your response.*