Let f be continuous, increasing and concave down on the interval $[1,\infty)$ as shown in Figure 1.

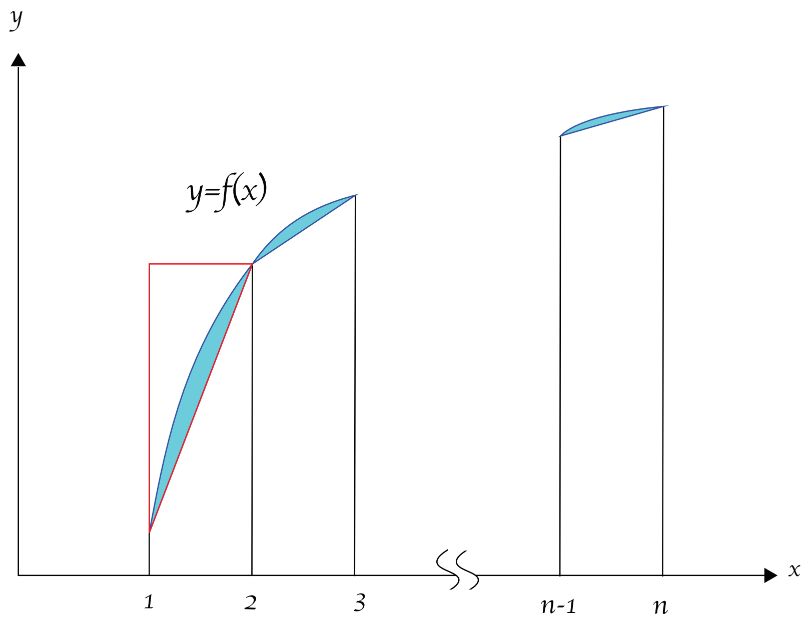

Figure 1: A continuous function f which is increasing and concave down.

For each natural number n, we let A_n be the total area of the regions shown in blue in the figure. This area A_n is formed by taking the area under the graph of f and subtracting the area of the trapezoids (drawn in the figure) beneath the graph from 1 to n. For example, noting that the area of a trapezoid is the base times average of the side lengths (heights), we have

$$\begin{aligned} A_3 &= \int_1^3 f(x)\,dx - 1\cdot\frac{f(1)+f(2)}{2} - 1\cdot\frac{f(2)+f(3)}{2} \\ &= \int_1^2 f(x)\,dx - \frac{f(1)+f(2)}{2} + \int_2^3 f(x)\,dx - \frac{f(2)+f(3)}{2} \\ &= \sum_{k=1}^{2}\left(\int_k^{k+1} f(x)\,dx - \frac{f(k)+f(k+1)}{2}\right). \end{aligned}$$

1. State the general formula for the sequence A_n.

2. Our assumption that f is concave down (look at Figure 1) means that the secant line between any two points on the graph of f lies (except for its endpoints) entirely beneath the graph of f. As a consequence, for each $k = 1, 2, \ldots, n$,
$$(1-t)f(k) + tf(k+1) < f(t+k) \tag{1}$$
for all $0 < t < 1$. Use properties of the integral (monotonicity of the integral[1]) to show that, for each $k = 1, 2, \ldots, n$,
$$\frac{f(k)+f(k+1)}{2} < \int_k^{k+1} f(x)\,dx.$$

[1]That is, if $f(x) < g(x)$ for all $a \le x \le b$, then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$.

Hint: Integrate the inequality (1) with respect to t from $t = 0$ to $t = 1$; the integer k should be treated as a constant. For your integral on the right hand side, make the change of variables $x = t + k$.

3. Use your results of the previous two parts to conclude that the terms (summands) of A_n are positive. Explain why this guarantees that the $\{A_n\}$ is an increasing sequence.

4. By a careful study of the geometry in Figure 1, it can be shown that, for each $n = 1, 2, 3 \ldots$, A_n is bounded above by T where T is the area of the red triangle in the figure. In other words,

$$A_n \leq T = \frac{f(2) - f(1)}{2} \tag{2}$$

for each $n = 1, 2, 3, \ldots$. Use Figure 2 below to explain why (2) is true, in your own words. You will need to explain the following:

- why each blue sliver can be moved into the red triangle (*Hint: use the concavity property to explain why each secant line may be moved into the red triangle so that the secant line will not intersect the graph of $f(x)$*)
- why no two blue slivers will intersect each other in the red triangle, except at the their common endpoint.
- why no blue sliver will intersect the top of the red triangle. (*Hint: use the increasing and concavity assumption.*)

You may find it useful to think about moving the second blue sliver into the red triangle and how this construction extends to the remaining blue slivers.

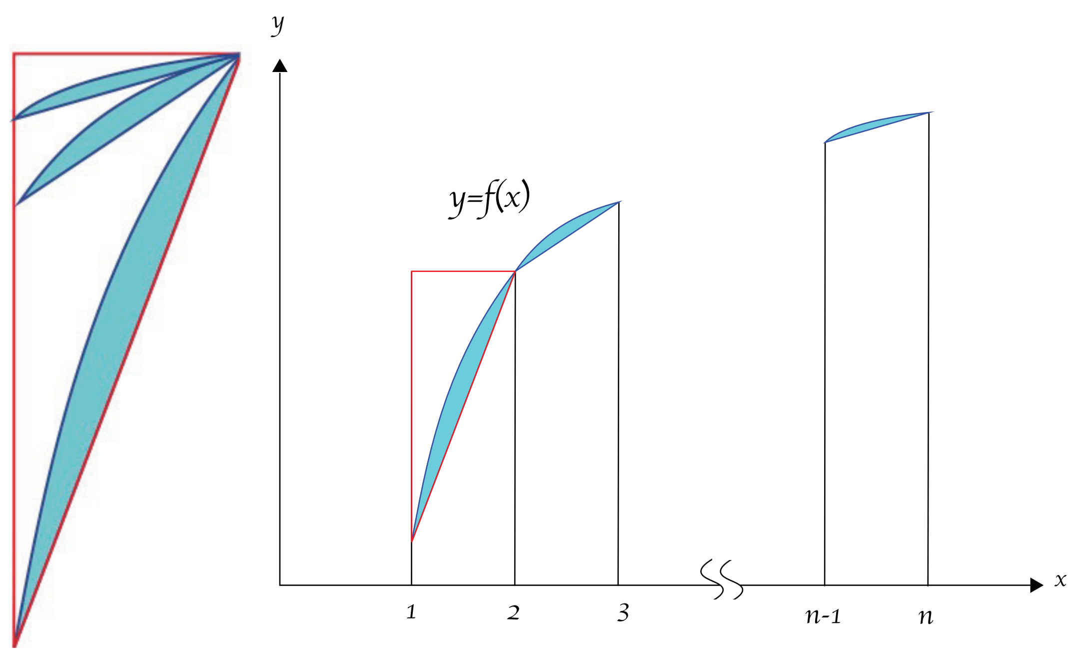

Figure 2: Sliding Slivers

5. Use the Monotonic Bounded Convergence Theorem, Theorem 3 in Section 11.1, to conclude that $\{A_n\}$ converges to some limit $K = \lim_{n\to\infty} A_n$.

6. Now, we focus on a special case. Let $f(x) = \ln x$ for $x \geq 1$. You may take for granted that this is continuous, increasing and concave down on $[1, \infty)$. It is also useful to remember that $x \ln x - x$ is an antiderivative of

$\ln x$; this can be found by a standard application of integration by parts. Show that

$$\begin{aligned} A_n &= \int_1^n \ln x\,dx - \left(\frac{\ln 1}{2} + \ln 2 + \cdots + \ln(n-1) + \frac{\ln n}{2}\right) \\ &= n\ln n - n + 1 - \ln n! + \ln\sqrt{n} \\ &= 1 + \ln\left(\frac{(n/e)^n\sqrt{n}}{n!}\right). \end{aligned}$$

Use Item 5 to conclude that $\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{n}} = e^{1-K}$.

In fact, it can be shown that $\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{n}} = \sqrt{2\pi}$. That is, $\lim A_n = K = 1 - \ln(\sqrt{2\pi})$ (*You do not have to show this!*)

7. From your result above, this means that

$$n! \approx \sqrt{2\pi n}(n/e)^n$$

for large n; this is called Stirling's Formula and it has many applications in mathematics, physics and computer science.

a. Use the formula to approximate 15! and check with your calculator that it's a decent approximation.

b. The number of digits of an integer x is

$$\log_{10}(x) + 1 = \frac{\ln(x)}{\ln(10)} + 1.$$

Use a calculator (e.g. Google or Wolfram Alpha) and Stirling's Formula to approximate the number of digits of 100!. To provide an appreciation of how large 100! is, the (conjectured) number of particles in the known universe is a number with ≈ 87 digits. Using Wolfram Alpha, you can calculate the number of digits of 100! exactly. How close is your approximation from the actual answer?