GEORGE'S SOLUTIONS (1)

Middlebury College

Calculus II: Fall 2017

September 11 Worksheet

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Let $f(x)$ be a continuous function, where $a \le x \le b$. Define the n^{th} *Riemann sum* to be the real number

$$S_n \stackrel{def}{=} \frac{(b-a)}{n}\left(f(x_1)+f(x_2)+f(x_3)+\ldots+f(x_{n-1})+f(x_n)\right),$$

where

$$\begin{aligned} x_1 &= a+\tfrac{1}{n}(b-a), \\ x_2 &= a+\tfrac{2}{n}(b-a), \\ x_3 &= a+\tfrac{3}{n}(b-a), \\ &\vdots \\ x_{n-1} &= a+\tfrac{(n-1)}{n}(b-a), \\ x_n &= a+\tfrac{n}{n}(b-a) \end{aligned}$$

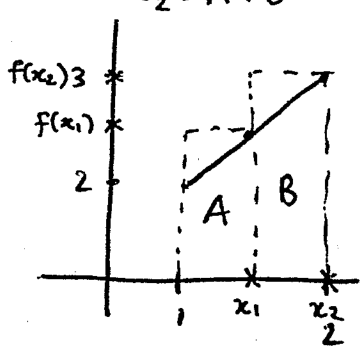
1. Consider the function $f(x) = x+1$, where $1 \le x \le 2$. In this exercise you will investigate the behaviour of S_n as n gets very large.

   (a) What are a, b in our setting?

   $a = 1 \qquad b = 2$

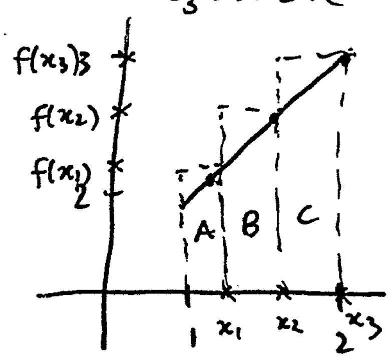
   (b) Draw three copies of the the graph of $f(x)$ below.

$S_2 = A + B$



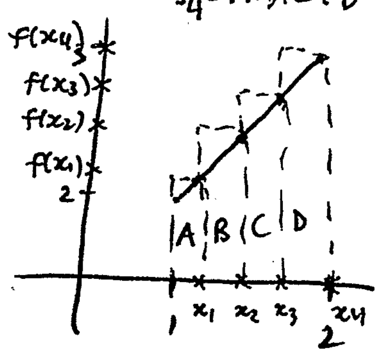
$n = 2$

$S_3 = A + B + C$



$n = 3$

$S_4 = A + B + C + D$



$n = 4$

   (c) For each $n = 2, 3, 4$, determine $x_1, \ldots x_n$ and plot the points $(x_1, 0), \ldots, (x_n, 0)$ on the graphs above.

   (d) For each $n = 2, 3, 4$, determine the n^{th} Riemann sum S_n. How can you intepret S_n using the graphs above? (*Hint: think rectangularly!*)

c) $(n=2)$

$x_1 = 1 + \frac{1}{2}(2-1) = \boxed{\frac{3}{2}}$

$x_2 = 1 + \frac{2}{2}(2-1) = \boxed{2}$

$(n=3)$

$x_1 = 1 + \frac{1}{3}(2-1) = \boxed{\frac{4}{3}}$

$x_2 = 1 + \frac{2}{3}(2-1) = \boxed{\frac{5}{3}}$

$x_3 = 1 + \frac{3}{3}(2-1) = \boxed{2}$

$(n=4)$

$x_1 = 1 + \frac{1}{4}(2-1) = \boxed{5/4}$

$x_2 = 1 + \frac{2}{4}(2-1) = \boxed{6/4}$

$x_3 = 1 + \frac{3}{4}(2-1) = \boxed{7/4}$

$x_4 = 1 + \frac{4}{4}(2-1) = \boxed{2}$

d)

$S_2 = \frac{2-1}{2}\left(f(\tfrac{3}{2}) + f(2)\right) = \frac{1}{2}\left(\frac{5}{2} + \frac{6}{2}\right) = \boxed{\frac{11}{4}}$

$S_3 = \frac{2-1}{3}\left(f(\tfrac{4}{3}) + f(\tfrac{5}{3}) + f(2)\right) = \frac{1}{3}\left(\frac{7}{3} + \frac{8}{3} + \frac{9}{3}\right) = \boxed{8/3}$

$S_4 = \frac{2-1}{4}\left(f(\tfrac{5}{4}) + f(\tfrac{6}{4}) + f(\tfrac{7}{4}) + f(\tfrac{8}{4})\right) = \frac{1}{4}\left(\frac{9}{4} + \frac{10}{4} + \frac{11}{4} + \frac{12}{4}\right)$

(e) What relationship do the real numbers S_n have in relation to each other? What is the relationship you expect S_5 to have with respect to S_2, S_3, S_4? Verify your prediction.

We observe that $S_2 > S_3 > S_4$

Expect: $S_5 < S_4$.

Check: $S_5 = \frac{1}{5}\left(\frac{11}{5} + \frac{12}{5} + \frac{13}{5} + \frac{14}{5} + \frac{15}{5}\right) = \frac{13}{5} < \frac{21}{8} = S_4$

(f) What do you expect to be the relation between the n^{th} Riemann sum S_n and the $(n+1)^{st}$ Riemann sum S_{n+1}? To what (real) number do you expect the real numbers S_n to approach as n gets very large? (*Hint: what happens to the rectangles as n increases?*)

Expect: $S_n > S_{n+1}$

Expect: S_n approximates area under graph of $f(x)$ and above x-axis $\Rightarrow$ S_n gets close to $\frac{5}{2}$

(g) (SPOT THE PATTERN!) Which of the following expressions gives the correct general formula for the n^{th} Riemann sum S_n?

i. $S_n = \frac{1}{n}\left(\frac{n+1}{n} + \frac{n+2}{n} + \frac{n+3}{n} + \ldots + \frac{n+(n-1)}{n} + \frac{n+n}{n}\right)$

ii. $S_n = \frac{1}{n}\left(\frac{2n+1}{n} + \frac{2n+2}{n} + \frac{2n+3}{n} + \ldots + \frac{2n+(n-1)}{n} + \frac{2n+n}{n}\right)$ ✓

iii. $S_n = \frac{1}{n}(1 + 2 + 3 + \ldots + (n-1) + n)$.

iv. none of the above.

(h) Using your choice of general formula above, can you say what happend to S_n as n gets very large? (*The formula* $1 + 2 + \ldots + (n-1) + n = \frac{1}{2}n(n+1)$ *may be useful.*)

$$S_n = \frac{1}{n}\left(\frac{2n+1}{n} + \frac{2n+2}{n} + \ldots + \frac{2n+(n-1)}{n} + \frac{2n+n}{n}\right)$$

$$= \frac{1}{n}\left(\underbrace{2+\frac{1}{n}}_{1} + \underbrace{2+\frac{2}{n}}_{2} + \ldots + \underbrace{2+\frac{n-1}{n}}_{n-1} + \underbrace{2+\frac{n}{n}}_{n}\right)$$

$$= \frac{1}{n}\left(\underbrace{(2+2+\ldots+2+2)}_{n \text{ TIMES}} + \frac{1}{n} + \frac{2}{n} + \ldots + \frac{n-1}{n} + \frac{n}{n}\right)$$

$$= \frac{1}{n}\left(2n + \frac{1+2+3+\ldots+(n-1)+n}{n}\right)$$

$$= \frac{1}{n}\left(2n + \frac{\frac{1}{2}\cancel{n}(n+1)}{\cancel{n}}\right), \text{ using formula above}$$

$$= 2 + \frac{1}{2}\frac{(n+1)}{n}$$

$$= 2 + \frac{1}{2} + \frac{1}{2n}$$; as n gets very large $\frac{1}{2n}$ gets very close to 0

$\Rightarrow S_n \to 2 + \frac{1}{2} = 5/2$

2. In this exercise you will investigate the construction of the *Koch snowflake* and try to determine its area[1].

We are going to describe a *recursive* procedure to define a planar region K known as the Koch snowflake. Each stage of the recursion is called an iteration. At the start of the recursion, the $n = 0$ iteration, we define $K(0)$ to be an equilateral triangle (say the sides all have length $1m$).

The next iteration of the recursive process creates the *snowflake* $K(1)$ by altering each perimeter line segment of the original triangle $K(0)$ as follows:

(i) Divide the line segment into three segments of equal length.

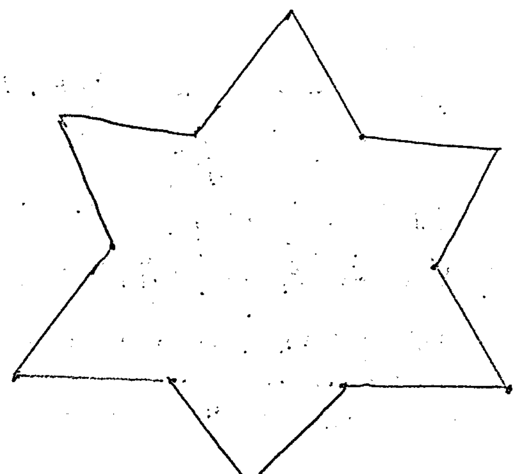
(ii) Draw an equilateral triangle that has the middle segment from the previous step as its base and points outward.

(iii) Remove the line segment that is the base of this newly created triangle

Having completed the above steps you have constructed the $n = 1$ snowflake $K(1)$.

(a) Follow the above procedure to construct the snowflake $K(1)$. Draw your snowflake below.

NOTE: area of an equilateral triangle having side length a is $\frac{\sqrt{3}}{4}a^2$



(b) Determine the area of $K(0)$ and $K(1)$

Area of $K(0)$ = area of equilateral triangle having side length $1 = \frac{\sqrt{3}}{4}$

Area of $K(1)$ = area of $K(0)$ + 3. (area of smaller triangle added to $K(0)$)

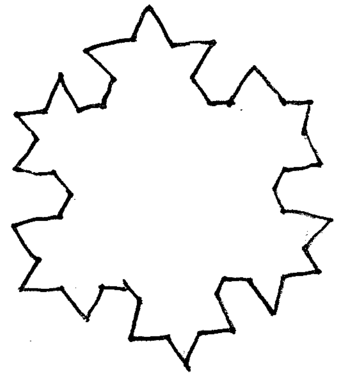
$$= \frac{\sqrt{3}}{4} + \underset{\substack{\uparrow \\ \text{no. of new triangles}}}{3} \cdot \left(\frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2\right) = \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{\sqrt{3}}$$

(c) Repeat steps (i)-(iii) for each perimeter line segment of $K(1)$ above to complete the second iteration of the recursive process. Once completed you have created the snowflake $K(2)$.

[1]This exercise is adapted from a similar exercise developed at Rockhurst University.

K(2) 4

Determine the area of $K(2)$.



Area of $K(2)$ =

$$\frac{\sqrt{3}}{4}+3\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)^2 + 12\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3^2}\right)^2 = \left\{ \begin{array}{l} \text{Area of } K(1) + \\ \quad 12 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{3^2}\right)^2 \end{array} \right.$$

no. of new triangles

$\frac{1}{3^2}=\frac{1}{3}\cdot\frac{1}{3}$ = side length of small triangles added to $K(1)$

(d) (SPOT THE PATTERN!) Having completed two iterations of the recursive process you have created two snowflakes $K(1), K(2)$. We could continue and perform the third iteration to create a third snowflake $K(3)$ by applying the steps (i)-(iii) to each perimeter line segment of $K(2)$ (although, depending on the size of your original triangle, this may be difficult to draw!). We are interested in determining the area of $K(3)$ *using the power of our minds!* Can you spot any patterns from your previous work that would help you predict the area of $K(3)$? Think of as many patterns as possible and write them down. **WAIT FOR FURTHER INSTRUCTION!**

(e) Predict the area that we add on to the area of $K(2)$ to obtain the area of $K(3)$.

Add on $48 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{3^3}\right)^2$ $\quad \frac{1}{3^3}=\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}$ = side length of small triangles added to $K(2)$ to form $K(3)$

no. of new triangles

(f) Predict the area we would add on to the area of $K(n)$ (the snowflake created by the n^{th} iteration of our process) to obtain the area of $K(n+1)$.

Add on: $\left(\text{no. of edges of } K(n)\right)\cdot\frac{\sqrt{3}}{4}\cdot\left(\frac{1}{3^{n+1}}\right)^2$

(g) Check your formula on the first three iterations of the recursive process. Use your formula to help you calculate the are of $K(4)$.

area of $K(4)$ = area of $K(3) + 192\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3^4}\right)^2$

(h) Express the total area of $K(4)$ as a sum of five terms. (*Hint: the first term should be the area of the original triangle $K(0)$*)

$$\text{Area of } K(4) = \frac{\sqrt{3}}{4} + 3\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)^2 + 12\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3^2}\right)^2 + 48\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3^3}\right)^2 + 192\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3^4}\right)^2$$

SUMMARY: (next week)

- You will show that:

$$\text{Area of } K(n) = \frac{1}{\sqrt{3}}\left(\frac{6-\left(\frac{2}{3}\right)^{2(n-1)}}{5}\right), \quad n \geq 1$$

- As n gets very large, $\left(\frac{2}{3}\right)^{2(n-1)}$ gets arbitrarily close to 0. Hence, as n gets very large

Area of $K(n)$ gets arbitrarily close to

$$\frac{1}{\sqrt{3}} \cdot \frac{6}{5} = \frac{2\sqrt{3}}{5}$$

i.e. the 'area' of the 'infinite' Koch snowflake $K(\infty) \overset{def}{=} \lim_{n\to\infty} K(n)$ is finite!

by definition, the snowflake we'd obtain if we continued our process AD INFINITUM.

- However, you will also show that the 'perimeter' of $K(\infty)$ has 'infinite' length.... ?!?

- This weird behaviour (finite area, infinite perimeter) is an exhibition of the fractal nature of $K(\infty)$