PROBLEM SET 5 : SOLUTIONS

A1 i)

a) Let $u = -x^2$

$$\frac{du}{dx} = -2x.$$

Since

$$x\exp(-x^2) = -\frac{1}{2}\exp(u)\frac{du}{dx},$$

the method of substitution gives

$$\int x\exp(-x^2)\,dx = \int -\frac{1}{2}\exp(u)\,du$$
$$= -\frac{1}{2}\exp(u) + C$$
$$= -\frac{1}{2}\exp(-x^2) + C.$$

b) Let $u = \log(x)$

$$\frac{du}{dx} = \frac{1}{x}.$$

Since $\frac{\log(x)}{x} = u\frac{du}{dx}$

the method of substitution gives

$$\int \frac{\log(x)}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C$$
$$= \frac{1}{2}(\log(x))^2 + C$$

c) Let $u = 1-x^2$

$$\frac{du}{dx} = -2x.$$

Since $x\sqrt{1-x^2} = -\frac{1}{2}\sqrt{u}\frac{du}{dx}$

the method of substitution gives

$$\int x\sqrt{1-x^2}\,dx = \int -\tfrac{1}{2}\sqrt{u}\,du$$

$$= -\tfrac{1}{3}u^{3/2} + C$$

$$= -\tfrac{1}{3}(1-x^2)^{3/2} + C.$$

d) Let $u = 1-x^2$

$$\frac{du}{dx} = -2x$$

Since $\dfrac{x}{\sqrt{1-x^2}} = -\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{u}}\dfrac{du}{dx}$

the method of substitution gives

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int -\frac{1}{2}\frac{1}{\sqrt{u}}\,du$$

$$= -\sqrt{u} + C$$

$$= -\sqrt{1-x^2} + C.$$

e) Note: $\log(\cos(x))\tan(x)$

$$= \log(\cos(x))\frac{\sin(x)}{\cos(x)}.$$

Let $u = \cos(x)$

$$\frac{du}{dx} = -\sin(x)$$

Since $\log(\cos(x))\tan(x) = -\dfrac{\log(u)}{u}\cdot\dfrac{du}{dx}$

the method of substitution gives

$$\int \log(\cos(x))\tan(x)\,dx$$

(b) above.

$$= \int -\frac{\log(u)}{u}\,du \quad = \frac{1}{2}(\log(u))^2 + C$$

$$= \frac{1}{2}(\log(\cos(x)))^2 + C$$

f) Let $u = \log(x)$

$$\frac{du}{dx} = \frac{1}{x}$$

Since

$$\frac{1}{x\log(x)} = \frac{1}{u}\cdot\frac{du}{dx}$$

the method of substitution gives

$$\int \frac{1}{x\log(x)}\,dx = \int \frac{1}{u}\,du$$

$$= \log(u) + C$$

$$= \log(\log(x)) + C$$

g) Let $u = \sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{x}}$$

Since

$$\frac{\exp(\sqrt{x})}{\sqrt{x}} = 2\exp(u)\frac{du}{dx}$$

the method of substitution gives

$$\int \frac{\exp(\sqrt{x})}{\sqrt{x}}\,dx = 2\int \exp(u)\,du$$

$$= 2\exp(u) + C$$

$$= 2\exp(\sqrt{x}) + C.$$

h) Let $u = \exp(x)$.

Then $\frac{du}{dx} = \exp(x)$.

Also: $\exp(2x) = \exp(x+x)$

$= \exp(x) \cdot \exp(x)$, by Remarkable Property

$= \exp(x)^2$

Hence,

$$\exp(2x) + 2\exp(x) + 1$$

$$= \exp(x)^2 + 2\exp(x) + 1$$

$$= (\exp(x) + 1)^2$$

Since

$$\frac{\exp(x)}{\exp(2x) + 2\exp(x) + 1} = \frac{1}{(u+1)^2}\frac{du}{dx}$$

the method of substitution gives

$$\int \frac{\exp(x)}{\exp(2x) + 2\exp(x) + 1}\,dx$$

$$= \int \frac{1}{(u+1)^2}\,du = -(u+1)^{-1} + C$$

$$= \frac{-1}{\exp(x)+1} + C.$$

(ii) a) Let $f = x$, $g' = \exp(x)$

$f' = 1$, $g = \exp(x)$

$$\int x\exp(x)\,dx = x\exp(x) - \int \exp(x)\,dx$$

$$= x\exp(x) - \exp(x) + C.$$

b) Let $f = x^2$ $\quad g' = x\exp(x^2)$

$f' = 2x$ $\quad g = \frac{1}{2}\exp(x^2)$

$$\int x^3\exp(x^2)\,dx = \frac{x^2}{2}\exp(x^2) - \int x\exp(x^2)\,dx$$
$$= \frac{x^2}{2}\exp(x^2) - \frac{1}{2}\exp(x^2) + C.$$

c) $I = \int \exp(x)\sin(2x)\,dx$

Let $f = \exp(x)$, $\quad g' = \sin(2x)$

$f' = \exp(x)$ $\quad g = -\frac{1}{2}\cos(2x)$

$$I = -\frac{1}{2}\cos(2x)\exp(x) + \frac{1}{2}\int \exp(x)\cos(2x)\,dx$$

Let $f = \exp(x)$ $\quad g' = \cos(2x)$

$f' = \exp(x)$ $\quad g = \frac{1}{2}\sin(2x)$

$$\Rightarrow I = -\frac{1}{2}\cos(2x)\exp(x) + \frac{1}{2}\left(\frac{1}{2}\exp(x)\sin(2x) - \frac{1}{2}\underbrace{\int \exp(x)\sin(2x)\,dx}_{=I}\right)$$

$$\Rightarrow I = -\frac{1}{2}\cos(2x)\exp(x) + \frac{1}{4}\exp(x)\sin(2x) - \frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = \frac{1}{2}\exp(x)\left(\frac{1}{2}\sin(2x) - \cos(2x)\right) + C$$

$$\Rightarrow I = \frac{2}{5}\exp(x)\left(\frac{1}{2}\sin(2x) - \cos(2x)\right) + C.$$

d) Let $f = (\log(x))^3$ $\quad g' = 1$

$f' = 3(\log(x))^2 \cdot \frac{1}{x}$ $\quad g = x$

$$\int (\log(x))^3 dx = x(\log(x))^3 - 3\int (\log(x))^2 dx$$

Let $f = (\log(x))^2$ $\quad g' = 1$

$f' = 2\log(x) \cdot \frac{1}{x}$ $\quad g = x.$

$$\int (\log(x))^2 dx = x(\log(x))^2 - 2\int \log(x)\, dx.$$

$$= x(\log(x))^2 - 2(\log(x)\cancel{x}\cancel{x} - x)$$

Hence;

$$\int (\log(x))^3 dx = x(\log(x))^3 - 3x(\log(x))^2 + 6\log(x)x - 6x$$

e) Let $f = \arctan(x)$ $\quad g' = 1$

$f' = \frac{1}{1+x^2}$ $\quad g = x.$

$$\int \arctan(x)\, dx = x\arctan(x) - \int \frac{x}{1+x^2} dx.$$

Let $u = 1+x^2$

$\frac{du}{dx} = 2x$

Since $\frac{x}{1+x^2} = \frac{1}{2}\frac{1}{u}\frac{du}{dx}$, by method of substitution,

$$\int \frac{x}{1+x^2} dx = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\log(u) + C$$

$$= \frac{1}{2}\log(1+x^2) + C$$

$\Rightarrow \int \arctan(x)\,dx$

$= x \arctan(x) - \frac{1}{2}\log(1+x^2) + C.$

f) Let $f = \log(x)$ $\quad g' = \sqrt{x}$

$f' = \frac{1}{x}$ $\quad g = \frac{2}{3}x^{3/2}$

$$\int \sqrt{x}\log(x)\,dx = \frac{2}{3}\log(x)x^{3/2} - \int \frac{2}{3}x^{1/2}\,dx$$

$$= \frac{2}{3}\log(x)x^{3/2} - \frac{4}{9}x^{3/2} + C.$$

A2 a) a) $\int \tan^2(x)\,dx = \int \frac{\sin^2(x)}{\cos^2(x)}\,dx$

$$= \int \frac{1-\cos^2(x)}{\cos^2(x)}\,dx$$

$$= \int \sec^2(x) - 1 \; dx$$

$$= \tan(x) - x + C.$$

b) $\int x\sin^2(x)\,dx$, $\quad \sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$= \int \frac{1}{2}(1-\cos(2x))x\,dx.$

$= \frac{1}{2}\int (x - x\cos(2x))\,dx$

Use integration by parts

$$\int x \cos(2x)\, dx \qquad f = x \quad g' = \cos(2x) \qquad f' = 1 \quad g = \tfrac{1}{2}\sin(2x)$$

$$= \frac{x}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\, dx$$

$$= \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)$$

Hence,

$$\int x \sin^2(x) = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x}{2}\sin(2x) - \frac{1}{4}\cos(2x)\right] + C.$$

c) $$\int (2-\sin(x))^2\, dx$$

$$= \int \left(4 - 4\sin(x) + \sin^2(x)\right) dx$$

$$= \int \left(4 - 4\sin(x) + \frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx$$

$$= 4x + 4\cos(x) + \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.$$

d) Let $u = \sin(x)$

$$\frac{du}{dx} = \cos(x).$$

Since $$\frac{\cos(x)}{\sin^2(x)} = \frac{1}{u^2}\cdot\frac{du}{dx}$$

by method of substitution

$$\int \frac{\cos(x)}{\sin^2(x)}\,dx = \int \frac{1}{u^2}\,du$$

$$= -u^{-1} + C$$

$$= \frac{-1}{\sin(x)} + C.$$

e) $\displaystyle\int \frac{1}{1-\sin(x)}\,dx.$

Observe:

$$\frac{1}{1-\sin(x)} = \frac{1}{1-\sin(x)} \cdot \frac{1+\sin(x)}{1+\sin(x)}$$

$$= \frac{1+\sin(x)}{1-\sin^2(x)}$$

$$= \int \frac{(1+\sin x)}{1-\sin^2(x)}\,dx$$

$$= \int \frac{1+\sin(x)}{\cos^2(x)}\,dx$$

$$= \int \sec^2(x) + \frac{\sin(x)}{\cos^2(x)}\,dx.$$

$$= \tan(x) + \sec(x) + C.$$

b) a) Let $x = \sin(t)$

$$\frac{dx}{dt} = \cos(t).$$

$$x^3\sqrt{1-x^2}\,\frac{dx}{dt} = \sin^3(t)\sqrt{1-\sin^2(t)}\cdot\cos(t)$$

$$= \sin^3(t)\cos^2(t)$$

$$= \sin(t)\cdot(1-\cos^2(t))\cos^2(t).$$

By method of inverse trig sub.

$$\int x^3 \sqrt{1-x^2}\, dx = \int \sin(t)\left(1-\cos^2(t)\right)\cos^2(t)\, dt.$$

Let $u = \cos(t)$

$$\frac{du}{dt} = -\sin(t)$$

Since

$$\sin(t)\left(1-\cos^2(t)\right)\cos^2(t) = -(1-u^2)u^2 \cdot \frac{du}{dt}$$

by method of substitution

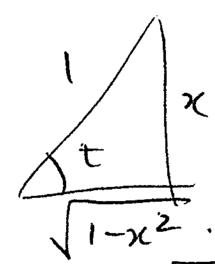
$$\int \sin(t)\left(1-\cos^2(t)\right)\cos^2(t)\, dt$$

$$= -\int (1-u^2)u^2\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5(t)}{5} - \frac{\cos^3(t)}{3} + C.$$

$$= \frac{(1-x^2)^{5/2}}{5} - \frac{(1-x^2)^{3/2}}{3} + C.$$



$\cos t = \sqrt{1-x^2}$

b) $\int \sqrt{1-4x^2}\, dx.$ Let $x = \frac{1}{2}\sin(t)$

$$\frac{dx}{dt} = \frac{1}{2}\cos(t)$$

$$\sqrt{1-4x^2} \cdot \frac{dx}{dt} = \sqrt{1-\sin^2(t)} \cdot \frac{1}{2}\cos(t)$$

$$= \frac{1}{2}\cos^2(t)$$

By method of inverse trig. substitution

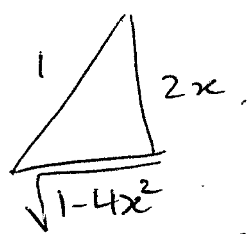
$$\int \sqrt{1-4x^2}\,dx = \frac{1}{2}\int \cos^2(t)\,dt$$

$$= \frac{1}{2}\int \left(\frac{1}{2} + \frac{1}{2}\cos(2t)\right)dt$$

$$= \frac{1}{4}\left[t + \frac{\sin(2t)}{2}\right] + C$$

$\sin(2t)$
$= 2\sin(t)\cos(t)$

$$\to = \frac{1}{4}\left[t + \sin(t)\cos(t)\right] + C.$$



$$= \frac{1}{4}\left[\arcsin(2x) + 2x\sqrt{1-4x^2}\right] + C$$

$\Rightarrow \cos(t) = \sqrt{1-4x^2}$

c) Let $x = \tan(t)$

$$\frac{dx}{dt} = \sec^2(t)$$

$$\frac{\sqrt{1+x^2}}{x}\cdot\frac{dx}{dt} = \frac{\sqrt{1+\tan^2(t)}}{\tan(t)}\cdot\sec^2(t)$$

$$= \frac{\sec^3(t)}{\tan(t)}, \qquad 1+\tan^2(t) = \sec^2(t)$$

$$= \frac{(1+\tan^2(t))\sec(t)}{\tan(t)}$$

$$= \frac{\sec(t)}{\tan(t)} + \tan(t)\sec(t)$$

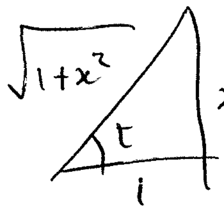
$$= \csc(t) + \tan(t)\sec(t).$$

By method of inverse trig substitution

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \csc(t) + \tan(t)\sec(t)\,dt$$

$x = \tan(t)$

$$= \log\left(\csc(t) - \cot(t)\right) + \sec(t) + C$$



$$= \log\left(\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right) + \sqrt{1+x^2} + C.$$

d) Write $x^2 + x + 1 = (x + \tfrac{1}{2})^2 + 3/4$

Let $x + \tfrac{1}{2} = \frac{\sqrt{3}}{2}\tan(t)$

$$\frac{dx}{dt} = \frac{\sqrt{3}}{2}\sec^2(t).$$

$$\frac{x}{\sqrt{x^2+x+1}}\,\frac{dx}{dt} = \frac{\frac{\sqrt{3}}{2}\tan(t) - \tfrac{1}{2}}{\sqrt{\frac{3}{4}\tan^2(t) + 3/4}} \cdot \frac{\sqrt{3}}{2}\sec^2(t)$$

$$= \left(\frac{\sqrt{3}}{2}\tan(t) - \tfrac{1}{2}\right) \cdot \sec(t).$$

$$= \frac{\sqrt{3}}{2}\sec(t)\tan(t) - \frac{1}{2}\sec(t)$$

By method of inverse trig substitution

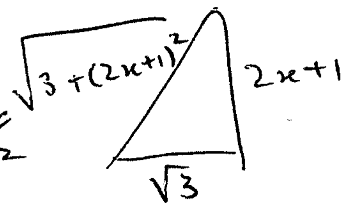
$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{\sqrt{3}}{2}\sec(t)\tan(t) - \tfrac{1}{2}\sec(t)\,dt.$$

$$= \frac{\sqrt{3}}{2}\sec(t) - \frac{1}{2}\int \sec(t)\,dt.$$

$$= \frac{\sqrt{3}}{2}\sec(t) - \frac{1}{2}\log\left(\sec(t) + \tan(t)\right) + C.$$

$$x + \tfrac{1}{2} = \frac{\sqrt{3}}{2}\tan(t)$$

$$\Rightarrow \frac{2x+1}{\sqrt{3}} = \tan(t)$$



$$\sqrt{3+(2x+1)^2} = \sqrt{4+4x+4x^2} = \sqrt{4(x^2+x+1)}$$

$$\Rightarrow \sec(t) = \frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\log\left(\frac{2}{\sqrt{3}}\sqrt{x^2+x+1} + \frac{2x+1}{\sqrt{3}}\right) + C.$$

e) Write $x^2+2x = (x+1)^2 - 1.$

Let $x+1 = \sec(t).$

$$\frac{dx}{dt} = \sec(t)\tan(t).$$

$$\begin{aligned}
\sqrt{x^2+2x}\cdot\frac{dx}{dt} &= \sqrt{\sec^2(t)-1}\cdot\sec(t)\tan(t) \\
&= \sqrt{\tan^2(t)}\cdot\sec(t)\tan(t) \\
&= \tan^2(t)\sec(t). \\
&= \frac{\sin^2(t)}{\cos^3(t)} \\
&= \frac{1-\cos^2(t)}{\cos^3(t)} \\
&= \sec^3(t) - \sec(t).
\end{aligned}$$

Hence, by method of trig substitution

$$\int \sqrt{x^2+2x}\, dx = \int \left(\sec^3(t) - \sec(t) \right) dt .$$

Let $f = \sec(t)$ $\qquad g' = \sec^2(t)$

$f' = \sec(t)\tan(t)$ $\qquad g = \tan(t)$

$$\int \sec^3(t) = \sec(t)\tan(t) - \int \sec(t)\tan^2(t)$$

$$= \sec(t)\tan(t) - \int \sec(\sec^2(t) - 1)\, dt .$$

$$= \sec(t)\tan(t) - \int \sec^3(t)\, dt + \int \sec(t)\, dt .$$

$$\Rightarrow \quad 2\int \sec^3(t)\, dt = \sec(t)\tan(t) + \int \sec(t)\, dt .$$

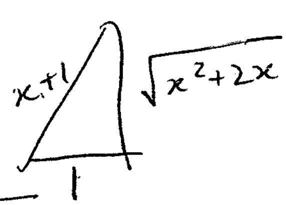
$$\Rightarrow \quad \int \sec^3(t)\, dt = \frac{1}{2} \sec(t)\tan(t) + \frac{1}{2}\int \sec(t)\, dt .$$

$$\Rightarrow \quad \int \sec^3(t)\, dt - \int \sec(t)\, dt .$$

$$= \frac{1}{2} \sec(t)\tan(t) - \frac{1}{2}\int \sec(t)\, dt$$

$$= \frac{1}{2} \sec(t)\tan(t) - \frac{1}{2} \log(\sec(t) + \tan(t)) + C .$$

As $x+1 = \sec(t)$,



$$\Rightarrow \quad \tan(t) = \sqrt{x^2+2x} .$$

Hence,

$$\int \sqrt{x^2+2x}\, dx = \frac{1}{2}(x+1)\sqrt{x^2+2x} - \frac{1}{2}\log\left((x+1)+\sqrt{x^2+2x}\right) + C.$$

A3: a) $\cdot$ $I_n = \int x^n \exp(x) dx$

(i) $f = x^n \qquad g' = \exp(x)$

$f' = n x^{n-1} \qquad g = \exp(x)$

$$I_n = x^n \exp(x) - n \int x^{n-1} \exp(x) dx$$

$$= x^n \exp(x) - n I_{n-1}$$

(ii) $I_0 = \int \exp(x) dx = \exp(x) + C$

(iii) $I_3 = x^3 \exp(x) - 3 I_2$

$$= x^3 \exp(x) - 3\left(x^2 \exp(x) - 2 \cdot I_1\right)$$

$$= x^3 \exp(x) - 3x^2 \exp(x) + 6 I_1$$

$$= x^3 \exp(x) - 3x^2 \exp(x) + 6\left(x \exp(x) - I_0\right)$$

$$= x^3 \exp(x) - 3x^2 \exp(x) + 6x \exp(x) - 6\exp(x) + C.$$

(b) $J_n = \int \sin^n(x) dx$, $\quad \sin^n(x) = \sin^{n-1}(x)\sin(x)$

$f = \sin^{n-1}(x)$

$f' = (n-1)\sin^{n-2}(x) \cdot \cos(x)$

$g' = \sin(x)$

$g = -\cos(x)$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x) dx.$$

Note: $\sin^{n-2}(x)\cos^2(x)$

$$= \sin^{n-2}(x)(1-\sin^2(x))$$

$$= \sin^{n-2}(x) - \sin^n(x)$$

$$\rightarrow J_n = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x) - \sin^n(x)\, dx$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x) dx - (n-1)\int \sin^n(x) dx$$

$$\Rightarrow \quad J_n = -\sin^{n-1}(x)\cos(x) + (n-1)J_{n-2} - (n-1)J_n$$

$$\Rightarrow \quad nJ_n = -\sin^{n-1}(x)\cos(x) + (n-1)J_{n-2}$$

$$\Rightarrow \quad J_n = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}J_{n-2}$$

(ii) $$J_0 = \int dx = x + C$$

$$J_1 = \int \sin(x)\,dx = -\cos(x) + C$$

(iii) $$J_4 = -\tfrac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\overbrace{\left[-\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}J_0\right]}^{J_2}$$

$$= -\tfrac{1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x + C$$

<u>A4:</u>

a) Use $\cos^2(x) + \sin^2(x) = 1$

b) $$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

c) $$\int \tan^4(x)\sec^6(x)\,dx$$

$$= \int \tan^4(x)\sec^4(x)\cdot\sec^2(x)\,dx.$$

~~Alt~~ ~~use tan~~

$$= \int \tan^4(x)\left(1+\tan^2(x)\right)^2\sec^2(x)\,dx$$

Let $\quad u = \tan(x)$

$$\frac{du}{dx} = \sec^2(x)$$

Since

$$\tan^4(x)(1+\tan^2(x))^2 \sec^2(x)$$

$$= u^4 (1+u^2)^2 \cdot \frac{du}{dx}$$

the method of substitution gives

$$\int \tan^4(x)(1+\tan^2(x))^2 \sec^2(x)\,dx$$

$$= \int u^4 (1+u^2)^2\, du$$

$$= \int (u^4 + 2u^6 + u^8)\, du = \frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^5(x)}{5} + \frac{2\tan^7(x)}{7} + \frac{\tan^9(x)}{9} + C.$$

d) $\displaystyle\int \tan^7(x)\sec^3(x)\,dx$

$$= \int \tan^6(x)\sec^2(x) \cdot \tan(x)\sec(x)\,dx$$

$$= \int (\sec^2(x)-1)^3 \sec^2(x) \tan(x)\sec(x)\,dx$$

Let $\quad u = \sec(x)$

$$\frac{du}{dx} = \sec(x)\tan(x)$$

Since $\quad (\sec^2(x)-1)^3 \sec^2(x) \sec(x)\tan(x)$

$$= (u^2-1)^3 u^2 \cdot \frac{du}{dx}$$

The method of substitution gives

$$\int \tan^7(x)\sec^3(x)\,dx$$

$$= \int (u^2-1)^3 u^2\,du$$

$$= \int (u^6-3u^4+3u^2-1)\,u^2\,du$$

$$= \frac{u^9}{9} - \frac{3u^7}{7} + \frac{3u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^9(x)}{9} - 3\frac{\sec^7(x)}{7} + 3\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C.$$

e) $I_n = \int \tan^n(x)\,dx$,

$$= \int \tan^{n-2}(x)\sec^2(x)\,dx - \underbrace{\int \tan^{n-2}(x)\,dx}_{I_{n-2}}$$

$$\tan^n(x)$$
$$= \tan^{n-2}(x)\cdot\tan^2(x)$$
$$= \tan^{n-2}(x)\left(\sec^2(x)-1\right)$$
$$= \tan^{n-2}(x)\sec^2(x) - \tan^{n-2}(x).$$

Let $f = \tan^{n-2}$ $\quad g' = \sec^2(x)$

$f' = (n-2)\cdot\tan^{n-3}(x)\sec^2(x)$ $\quad g = \tan(x).$

$$= \left[\tan^{n-1}(x) - (n-2)\int \tan^{n-2}(x)\sec^2(x)\,dx\right] - I_{n-2}$$

$$\Rightarrow \quad I_n = \tan^{n-1}(x) - (n-2)\int \tan^{n-2}(x)\sec^2(x)\,dx$$

$$- \, I_{n-2}$$

Use $\sec^2(x) = \tan^2(x) + 1$,

$$I_n = \tan^{n-1}(x) - (n-2)\int \tan^{n-2}(x)\left(\tan^2(x)+1\right)dx$$

$$- I_{n-2}$$

$$= \tan^{n-1}(x) - (n-2)\int \tan^{n}(x) + \tan^{n-2}(x)\,dx$$

$$- \, I_{n-2}$$

$$= \tan^{n-1}(x) - (n-2) I_n - (n-1) I_{n-2}$$

$$\Rightarrow \quad (n-1) I_n = \tan^{n-1}(x) - (n-1) I_{n-2}$$

$$\Rightarrow \quad \boxed{I_n = \frac{1}{n-1}\tan^{n-1}(x) - I_{n-2}.}$$