Calculus II: Fall 2017
Problem Set 5

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Keywords: integration by parts, integration by substitution, inverse trigonometric substitution.

Problems for submission

A1. Determine the following integrals using the technique specified.

(i) *Integration by substitution*:

a) $\int x\exp(-x^2)dx$ b) $\int \frac{\log(x)}{x}dx$ c) $\int x\sqrt{1-x^2}dx$ d) $\int \frac{x}{\sqrt{1-x^4}}dx$

e) $\int \log(\cos(x))\tan(x)dx$ f) $\int \frac{1}{x\log(x)}dx$ g) $\int \frac{\exp(\sqrt{x})}{\sqrt{x}}dx$ h) $\int \frac{\exp(x)}{\exp(2x)+2\exp(x)+1}dx$

(ii) *Integration by parts*:

a) $\int x\exp(x)dx$ b) $\int x^3\exp(x^2)dx$ c) $\int \exp(x)\sin(2x)dx$

d) $\int (\log(x))^3dx$ e) $\int \arctan(x)dx$ f) $\int \sqrt{x}\log(x)dx$

A2. (a) Determine the following trigonometric integrals.

a) $\int \tan^2(x)dx$ b) $\int x\sin^2(x)dx$ c) $\int (2-\sin(x))^2dx$ d) $\int \frac{\cos(x)}{\sin^2(x)}dx$ e) $\int \frac{1}{1-\sin(x)}dx$

(b) Determine the following integrals using inverse trigonometric substitution.

a) $\int x^3\sqrt{1-x^2}dx$ b) $\int \sqrt{1-4x^2}dx$ c) $\int \frac{\sqrt{1+x^2}}{x}dx$ d) $\int \frac{x}{\sqrt{x^2+x+1}}dx$ e) $\int \sqrt{x^2+2x}dx$

You will want to recall how to complete the square. Use the following inverse trigonometric substitutions

$$\begin{aligned} x = a\sin(t) &\quad\leftrightarrow\quad \sqrt{a^2-x^2} \\ x = a\tan(t) &\quad\leftrightarrow\quad \sqrt{a^2+x^2} \\ x = a\sec(t) &\quad\leftrightarrow\quad \sqrt{x^2-a^2} \end{aligned}$$

A3. In this problem you will investigate certain integration *reduction formulae*.

(a) Let $n \geq 0$ be an integer. Define

$$I_n = \int x^n\exp(x)dx$$

i. Use integration by parts to show the following reduction formula

$$I_n = x^n\exp(x) - nI_{n-1}$$

ii. Show that $I_0 = \exp(x)$.

iii. Use the reduction formula to compute

$$\int x^3\exp(x)dx$$

(*Hint: use repeated applications of the reduction formulae to write I_3 in terms of I_0*)

(b) Let $n \geq 0$ be an integer. Define

$$J_n = \int \sin^n(x)dx$$

i. Use integration by parts to show the following reduction formula

$$J_n = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}J_{n-2}$$

ii. Show that $J_0 = x$ and $J_1 = -\cos(x)$.

iii. Use the reduction formula to determine

$$\int \sin^4(x)dx$$

(*Hint: use repeated applications of the reduction formula to determine* J_4 *in terms of* J_0)

iv. Now, use the formula for $\sin^2(x)$ to determine $\int \sin^4(x)dx$. Deduce a (possibly) new trigonometric identity.

A4. In this problem you will investigate integrals of the form

$$\int \tan^m(x)\sec^n(x)dx$$

where $m, n \geq 0$ are integers.

(a) Explain why $\sec^2(x) = 1 + \tan^2(x)$.

(b) Determine $\frac{d}{dx}\tan(x)$ and $\frac{d}{dx}\sec(x)$.

(c) Use the method of substitution to determine

$$\int \tan^4(x)\sec^6(x)dx$$

(d) Use the method of substitution to determine

$$\int \tan^7(x)\sec^3(x)dx$$

(e) Let $n \geq 0$ be an integer. Determine the reduction formula

$$\int \tan^n(x)dx = \frac{1}{n-1}\tan^{n-1} - \int \tan^{n-2} dx$$

Use the reduction formula to determine $\int \tan^5(x)dx$ and $\int \tan^6(x)dx$.

Additional recommended problems (not for submission)

B1. Determine the following reduction formulae:

(a) Let $n \geq 0$ be an integer. Use integration by parts to determine the reduction formula

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$$

Use the reduction formula to compute $\int \cos^7(x)dx$.

(b) Let $n \geq 0$ be an integer. Use integration by parts to determine the reduction formula

$$\int (x^2+1)^n dx = \frac{x(x^2+1)^n}{2n+1} - \frac{2n}{2n+1}\int (x^2+1)^{n-1}dx$$

(*Hint: the identity* $x^2(x^2+1)^k = (x^2+1)^{k+1} - (x^2+1)^k$ *will be useful*)
Use the reduction formula to compute $\int (x^2+1)^4 dx$.

Challenging Problems

C1. (*)