PROBLEM SET 4

A1 a) Let $x \geq 0$. Then, the sequence $(S_m(x))_m$ is increasing:

$$S_1(x) = 1 + x$$
$$S_2(x) = 1 + x + \frac{x^2}{2!}$$
$$\vdots$$
$$S_m(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^m}{m!}$$

As $\exp(x) = \lim_{m \to \infty} S_m(x)$, we must have

$S_m(x) \leq \exp(x)$ for all m.

b) Fix m. Consider

$$\frac{\exp(x)}{x^m} \geq \frac{S_{m+1}(x)}{x^m} = \frac{1 + x + \frac{x^2}{2!} + \dots + \frac{x^{m+1}}{(m+1)!}}{x^m}$$

$$= \overbrace{\frac{1}{x^m} + \frac{1}{x^{m-1}} + \frac{1}{2!\,x^{m-2}} + \dots + \frac{1}{m!}}^{>0} + \frac{x}{(m+1)!}$$

$$> \frac{x}{(m+1)!}, \quad b/c \quad x \geq 0$$

c) Since $\frac{x}{(m+1)!}$ is unbounded as x gets very large, and $\frac{\exp(x)}{x^m} > \frac{x^1}{(m+1)!}$ for all $x \geq 0$, we have $\frac{\exp(x)}{x^m}$ is unbounded as $x \to +\infty$.

A2 Let $f(x) = 1 + \sqrt{2+3x}$

a) i) $f(x)$ is defined whenever $2+3x \geq 0$

$$\Rightarrow \quad x \geq -\frac{2}{3}.$$

ii) Since $\sqrt{2+3x} \geq 0$ we have

$$f(x) \geq 1 \, , \text{ for all } x.$$

If $c \geq 1$ then

$$c = f\left(\tfrac{1}{3}((c-1)^2 - 2)\right)$$

Hence, the range is $y \geq 1$.

iii) If $x \neq y$ then

$$f(x) = 1 + \sqrt{2+3x} \neq 1 + \sqrt{2+3y} = f(y).$$

Hence, f is one-to-one.

(iv) Consider

$$y = f(x)$$
$$= 1 + \sqrt{2+3x}$$

$$\Rightarrow \quad (y-1)^2 = 2+3x$$

$$\Rightarrow \quad \frac{1}{3}\left[(y-1)^2 - 2\right] = x.$$

Hence, $f^{-1}(y) = \frac{1}{3}\left((y-1)^2 - 2\right)$.

A2 b) (i)

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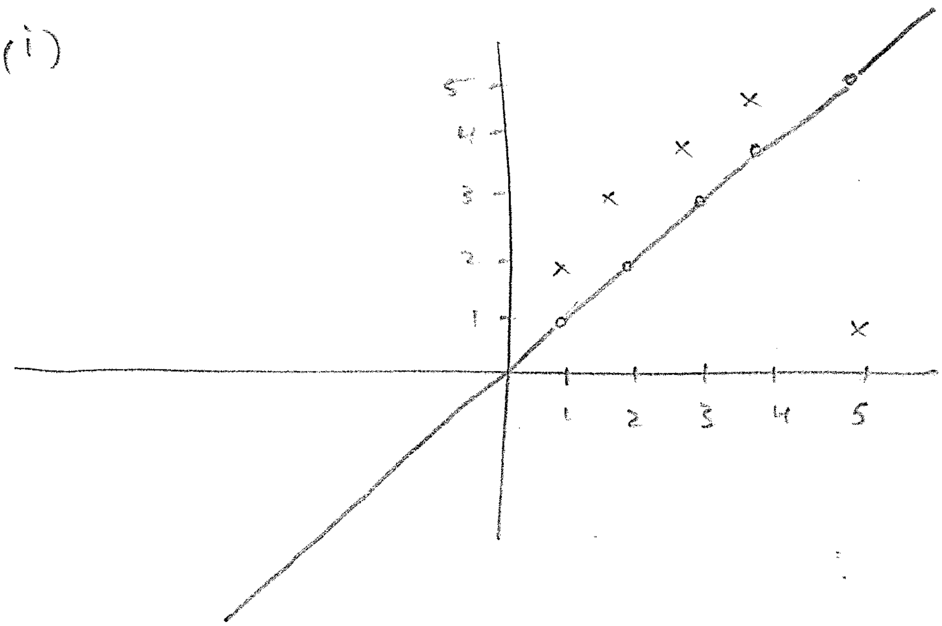
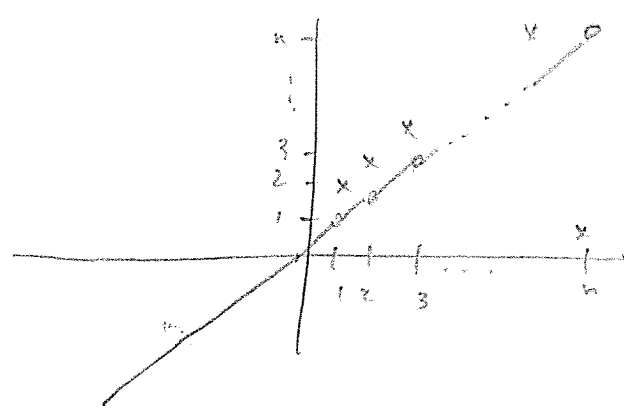


(ii) Graph of $g_5(x)$ passes horizontal line test. We have

$$g_5^{-1}(y) = \begin{cases} y & y \neq 1,2,3,4,5 \\ y-1 & y = 2,3,4,5 \\ 5 & y = 1. \end{cases}$$

= " unique x-value s.t. $g_5(x) = y$

iii) Similarly, the graph of $g_n(x)$ is



Passes horizontal line test.

We have

$$g_n^{-1}(y) = \begin{cases} y & y \neq 1, 2, \ldots, n \\ y-1 & y = 2, 3, \ldots, n \\ n & y = 1. \end{cases}$$

A3 a) $\frac{d}{dx} f(x)$

$$= \frac{d}{dx} \log(xc)$$

$$= \frac{1}{xc} \cdot c = \frac{1}{x}.$$

b) Using Proposition 3.1 from October 12 Lecture there is K, such that

$$f(x) = \log(x) + K.$$

c) Let $x = 1$. Then.

$$\log(c) = f(1)$$
$$= \log(1) + K$$
$$= 0 + K, \quad \text{b/c } \log(1) = 0.$$
$$= K,$$

Hence, for any $x, c > 0$.

$$\log(xc) = f(x) = \log(x) + \log(c).$$

d) Let $x > 0$. Then,

$$0 = \log(1)$$
$$= \log(x \cdot x^{-1})$$
$$= \log(x) + \log(x^{-1}) \text{ , by (c)}$$

$$\Rightarrow -\log(x) = \log(x^{-1}).$$

e) Let $P(n)$: $\log(x^n) = n \log(x)$.

Base case: $P(1)$

$\log(x) = 1 \log(x)$ ✓

Induction step: Assume $P(k)$. $\log(x^k) = k \log(x)$.

want to show

$P(k+1)$: $\log(x^{k+1}) = (k+1) \log(x)$

Now,

$$\log(x^{k+1}) = \log(x^k \cdot x)$$
$$= \log(x^k) + \log(x)$$
$$= k \log(x) + \log(x)$$
$$= (k+1) \log(x).$$

Hence, $P(k+1)$.

So, by induction $P(n)$ for all n.

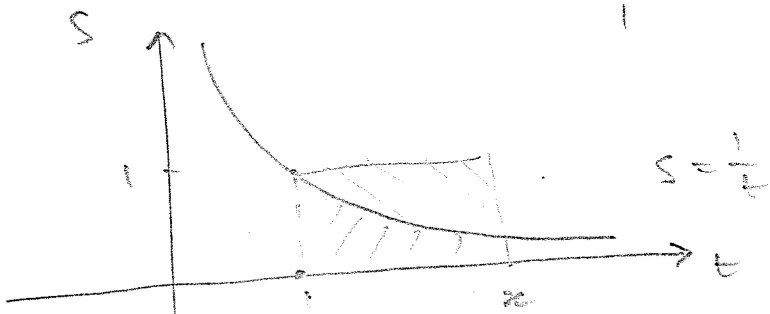
If $n<0$, then $n=-m$, $m>0$, and by what we've just shown,

$$\log(x^n) = \log(x^{-m})$$
$$= \log((x^{-1})^m)$$
$$= m\log(x^{-1})$$
$$= m(-\log(x))$$
$$= (-m)\log(x)$$
$$= n\log(x).$$

Also, if $n=0$,

$$\log(x^0) = \log(1) = 0 \quad \checkmark.$$

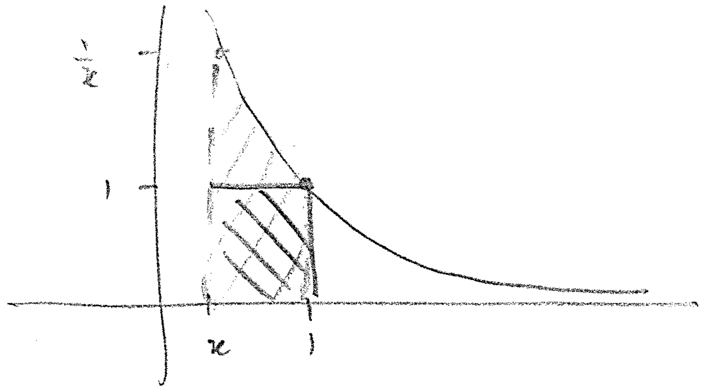
A4: We have $\log(x) = \int_1^x \frac{dt}{t}$.

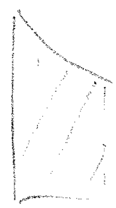


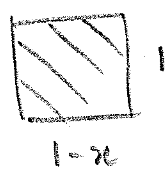
Spse $x \geq 1$. Then, $\log(x) =$ 

<  $= x-1$.

If $0<x<1$ then



$\log(x) = -$ 

$<\ -$  $= -1\cdot(1-x) = x-1.$

Hence, if $x \geqslant 2$ then

$$\log(x) < x-1$$
$$< (x-1)^k \quad \text{, for any natural number } k \geqslant 1$$