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Problems for submission

A1. In this problem you will show $\exp(x)$ grows faster than x^m, for any natural number m.

(a) Let m be a natural number. Show that

$$\exp(x) \geq s_m(x), \qquad \text{for all } x \geq 0.$$

Here $s_m(x)$ is the m^{th} partial sum of the series $1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$. (*Hint:* $\exp(x)$ *is the limit of the increasing sequence* $(s_m(x))$*.*)

(b) Let m be a natural number. By considering $\frac{s_{m+1}(x)}{x^m}$, show that

$$\frac{\exp(x)}{x^m} \geq \frac{x}{(m+1)!}, \qquad \text{for all } x \geq 0.$$

(c) Explain why

$$\frac{\exp(x)}{x^m}$$

is unbounded as $x \to \infty$. Deduce that $\exp(x)$ grows faster than x^m, for any m.

A2. (a) Let $f(x) = 1 + \sqrt{2 + 3x}$.

i. Determine the domain A of $f(x)$.
ii. Determine the range B of $f(x)$.
iii. Explain why $f(x)$ is one-to-one.
iv. Determine the inverse function $f^{-1}(y)$, taking care to describe its domain and range.

(b) Let n be a natural number. Consider the function

$$g_n(x) = \begin{cases} x, & \text{if } x \neq 1, 2, 3, \dots, n, \\ x + 1, & \text{if } x = 1, 2, \dots, n-1, \\ 1, & \text{if } x = n. \end{cases}$$

i. Draw the graph of $g_5(x)$. What is its domain and range?
ii. Show that $g_5(x)$ is one-to-one. Determine the inverse function $g_5^{-1}(y)$.
iii. Show that $g_n(x)$ is one-to-one. Determine the inverse function $g_n^{-1}(y)$.

A3. In this problem you will determine the standard logarithm rules using the definition of $\log(x)$ as an antiderivative of $1/t$.

Recall that $\log(x) = \exp^{-1}(x)$, the inverse function of $\exp(x)$, was determined (October 12 Lecture) to be the function

$$\log(x) \stackrel{def}{=} \int_1^x \frac{dt}{t}$$

(a) Let $c > 0$ be a real number. Define $f(x) = \log(xc)$. Using the Fundamental Theorem of Calculus and the chain rule, show that

$$f'(x) = \frac{1}{x}.$$

(b) Deduce that there is some constant K such that $f(x) = \log(x) + K$. (*Hint: if $g(x)$ and $h(x)$ are two functions such that $g'(x) = h'(x)$ then $g(x) = h(x) + K$, for some constant K*)

(c) Show that $K = \log(c)$. (*Hint: consider $f(1)$*). Deduce that

$$\log(xc) = \log(x) + \log(c) \qquad (*)$$

for any $x, c > 0$.

(d) Let $x > 0$. By considering the fact that $1 = x \cdot x^{-1}$, use $(*)$ to show that

$$\log(x^{-1}) = -\log(x).$$

(e) Let $x > 0$. Using induction show that

$$\log(x^n) = n\log(x), \qquad n = 1, 2, 3, \ldots$$

Deduce that

$$\log(x^n) = n\log(x)$$

for any integer n (not necessarily positive).

A4. Show that

$$\log(x) < x - 1, \qquad \text{for any } x > 0.$$

Deduce that $\log(x) < x^k$, for any natural number k and any $x > 2$.

Additional recommended problems (not for submission)

B1. Let $f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}$.

(a) Show that $f(x) = \frac{2}{1+\sqrt{x}} - 1$.

(b) Determine the domain A of $f(x)$.

(c) Show that the range B of $f(x)$ is the collection of all real numbers $-1 < y \leq 1$.

(d) Explain why $f(x)$ is strictly decreasing. Deduce that $f(x)$ is one-to-one.

(e) Determine the inverse function $f^{-1}(y)$.

B2. (a) Verify the following identities

$$a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3),$$

$$a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$$

(b) Let a, b be real numbers. Using induction show that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}), \quad n = 2, 3, 4, \ldots$$

Hint: $n = 2$ is the base case. For the inductive step consider

$$a^{n+1} - b^{n+1} = a^{n+1} - ab^n + ab^n - b^{n+1}$$

B3. Define

$$\sinh(x) = \frac{1}{2}\left(\exp(x) - \exp(-x)\right), \quad \text{and} \quad \cosh(x) = \frac{1}{2}\left(\exp(x) + \exp(-x)\right)$$

. We call $\sinh(x)$ (pronounced *sinch of x*) the **hyperbolic sine function** and $\cosh(x)$ the **hyperbolic cosine function**.

(a) Show that $\sinh(-x) = -\sinh(x)$ and $\cosh(-x) = \cosh(x)$.

(b) Show that $\cosh^2(x) - \sinh^2(x) = 1$.

(c) Show that

$$\frac{d}{dx}\sinh(x) = \cosh(x), \qquad \frac{d}{dx}\cosh(x) = \sinh(x).$$

(d) Use the derivative of $\sinh(x)$ to explain why it is a strictly increasing function. Deduce that $\sinh(x)$ is one-to-one.

(e) The domain and range of $\sinh(x)$ is the collection of all real numbers. Hence, its inverse function $\sinh^{-1}(y)$ also has its domain and range being the collection of all real numbers. You will now show that

$$\sinh^{-1}(y) = \log(y + \sqrt{y^2+1}).$$

i. Let $x = \sinh^{-1}(y)$. Show that

$$\exp(x) - 2y - \exp(-x) = 0.$$

Deduce that

$$\exp(x)^2 - 2y\exp(x) - 1 = 0.$$

ii. Use the quadratic formula to show that

$$\exp(x) = y \pm \sqrt{y^2+1}$$

iii. Show that $\exp(x) = y + \sqrt{y^2+1}$. (*Hint:* $\exp(x) > 0$ *always. Now, consider* $y - \sqrt{y^2+1}$)

iv. Deduce that

$$\sinh^{-1}(y) = \log(y + \sqrt{y^2+1})$$

Challenging Problems

C1. (*) In this problem you will show that Euler's number

$$e = \exp(1) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2.71828\ldots$$

is irrational using a *proof by contradiction* argument. We will <u>*assume*</u> that e is, in fact, a rational number (i.e. a ratio of two integers) and derive a statement of absurdity. Hence, it must be the case that e is irrational (any real number must be either rational *or* irrational).

Assume that e is rational. This means that there are two natural numbers p and q so that

$$e = \frac{p}{q}$$

We will assume that p, q satisfy the condition that p is not a multiple of q.

(a) Explain why $q \neq 1$. Deduce that $q > 1$. (*Hint: what would have to be true of* e *if* $q = 1$*?*)

(b) Let s_m be the m^{th} partial sum of the series $1 + \sum_{n=1}^{\infty}\frac{1}{n!}$. Show that $q!s_q$ is an integer. Deduce that

$$q!(e - s_q)$$

is an integer.

We will now show, by a different argument, that $q!(e - s_q)$ is *not* an integer. This contradiction of what we've just shown implies that our assumption that e is rational must be a false assumption. Hence, e can't possibly be rational, so it must be an irrational number.

Observe that, in the argument that follows, we never make use of our assumption that $e = p/q$.

(c) Using the definition of e as the limit of a series, show that

$$e - s_q = \sum_{n=q+1}^{\infty} \frac{1}{n!} = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \dots$$

(d) Deduce that

$$q!\,(e - s_q) = \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \dots$$

(e) Let

$$a_n = \frac{1}{(q+1)(q+2)\cdots(q+n)}.$$

Show that

$$a_n \le \frac{1}{(q+1)^n}, \qquad n = 1, 2, 3, \dots$$

Deduce that

$$q!(e - s_q) \le \sum_{n=1}^{\infty} \frac{1}{(q+1)^n} = \frac{1}{q} < 1.$$

(f) Recall from Problem A1 that $e > s_q$. Show that

$$0 < q!(e - s_q) < 1.$$

Conclude that $q!(e - s_q)$ can't possibly be an integer.

C2. (***)