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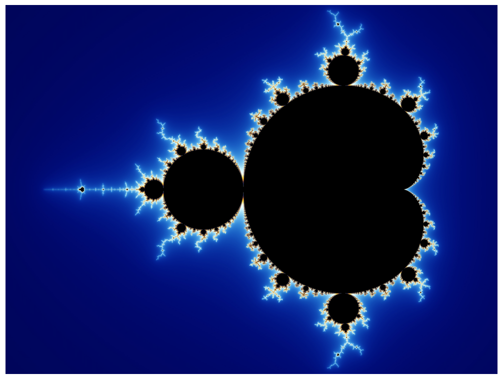
Problems for submission

A1. In this problem you will investigate the 'real version' of the Mandelbrot set: you will verify that a region in the following figure should be **black**.



The Mandelbrot set is a region of the complex plane described by the convergence properties of a family of sequences of complex numbers (z_n).

Let $r > 0$ and define the sequence $(a_n)_{n\geq 1}$, where

$$a_1 = 0, \qquad a_{n+1} = r + a_n^2, \quad n = 1, 2, 3, \ldots$$

(a) Suppose that the sequence (a_n) is convergent and denote $L = \lim_{n\to\infty} a_n$. Show that $L = r + L^2$ and

$$L = \frac{1}{2}\left(1 \pm \sqrt{1-4r}\right)$$

(*Hint: recall that* $\lim a_{n+1} = \lim a_n$)
Deduce that $0 < r \leq \frac{1}{4}$.

(b) *For the remainder of this problem we assume that* $0 < r \leq \frac{1}{4}$. Use mathematical induction to show that $a_n \geq 0$, for $n = 1, 2, 3, \ldots$.

(c) Use mathematical induction to show that $a_n \leq \frac{1}{2}$, for $n = 1, 2, 3, \ldots$.

(d) Use mathematical induction to show that $a_{n+1} \geq a_n$, for $n = 1, 2, 3, \ldots$.

(e) Explain carefully why (a_n) is convergent whenever $0 < r \leq \frac{1}{4}$.

A2. Use the sum of the first 10 terms to approximate the sum of the convergent series

$$L = \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Provide an estimate of the error $|L - s_{10}|$.

A3. Using the Root Test determine the values of x for which the given series converges and diverges.

a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{4^n}$ b) $\sum_{n=1}^{\infty} \frac{x^n}{(n+1)^3}$ c) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ d) $\sum_{n=1}^{\infty} \frac{(-x)^n}{n2^n}$

A4. Let (F_n) be the Fibonacci sequence, defined recursively as follows

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad n = 3, 4, 5, \dots$$

(a) Using induction, show that F_{3n} is an even integer, for every $n = 1, 2, 3, \dots$ (i.e. every third term of the Fibonacci sequence is even). Recall that an integer x is even if $x = 2y$, for some integer y.

(b) Using induction, show that $F_n < 2^n$, for every natural number n.

Additional recommended problems (not for submission)

B1. Prove that $n! > 2^n$, for every natural number n.

B2. (a) Using induction, prove that $n^2 < 3^n$, for $n = 2, 3, 4, \dots$. (*Hint: for the inductive step it might help to consider the parabola* $y = 2x^2 - 2x - 1$)

(b) Deduce that $n^2 < 3^n$, for every natural number n.

(c) Prove that $n^3 \leq 3^n$, for every natural number n.

B3. Let a, b be distinct integers. Prove that $a^n - b^n$ is divisible by $a - b$, for every $n = 1, 2, 3, \dots$.

B4. For which k is the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

convergent?

B5. Determine the mxaimum number of regions in which the plane can be divided by n straight lines, for every natural number n.

Challenging Problems

C1. (*) Divide the plane into regions using straight lines. Prove that those regions can be coloured with two colours so that no two regions that share a boundary have the same colour.

C2. (***) In this problem you will prove the **Riemann Rearrangement Theorem**:

Let $\sum a_n$ be a conditionally convergent series, r a real number. Then, there is a rearrangement (b_n) of the sequence (a_n) so that the series $\sum b_n$ converges to r.

Given a series $\sum a_n$ we define the series $\sum p_n$ whose terms (p_n) are all the positive terms of the sequence (a_n), and a series $\sum q_n$ whose terms (q_n) are all the negative terms of the sequence (a_n). Specifically,

$$p_n = \frac{a_n + |a_n|}{2}, \qquad q_n = \frac{a_n - |a_n|}{2}.$$

Observe that, if $a_n > 0$ then $p_n = a_n$ and $q_n = 0$, and if $a_n < 0$ then $q_n = a_n$ and $p_n = 0$.

(a) Suppose that $\sum a_n$ is absolutely convergent. Show that both of the series $\sum p_n$ and $\sum q_n$ are convergent.

(b) Suppose that $\sum a_n$ is conditionally convergent. Show that one of the series $\sum p_n$ or $\sum q_n$ must be divergent. Deduce that the corresponding sequence of partial sums is unbounded.

(c) Suppose that $\sum a_n$ is conditionally convergent. Show that *both* $\sum p_n$ and $\sum q_n$ must have unbounded sequences of partial sums.

(d) Let r be a real number.

i. Show that there exists N such that $\sum_{n=1}^{N} p_n > r$. Define N_1 to be the least natural number such that

$$S_1 \stackrel{def}{=} \sum_{n=1}^{N_1} p_n > r.$$

ii. Show that there exists M such that $\sum_{n=1}^{N_1} p_n + \sum_{n=1}^{M} q_n < r$. Define M_1 to be the least natural number such that

$$T_1 \stackrel{def}{=} \sum_{n=1}^{N_1} p_n + \sum_{n=1}^{M_1} q_n < r.$$

iii. Similarly, let $N_2 > N_1$ be the least natural number such that

$$S_2 \stackrel{def}{=} \sum_{n=1}^{N_2} p_n + \sum_{n=1}^{M_1} q_n > r.$$

Explain why N_2 exists.

iv. Similarly, let $M_2 > M_1$ be the least natural number such that

$$T_2 \stackrel{def}{=} \sum_{n=1}^{N_2} p_n + \sum_{n=1}^{M_2} q_n < r.$$

Explain why M_2 exists.

v. Continuing in this way, show that you can obtain an increasing sequence of integers

$$N_1 < N_2 < N_3 < \cdots \qquad M_1 < M_2 < M_3 < \cdots$$

and sums

$$S_k \stackrel{def}{=} \sum_{n=1}^{N_k} p_n + \sum_{n=1}^{M_{k-1}} q_n, \quad \text{and} \quad T_k \stackrel{def}{=} \sum_{n=1}^{N_k} p_n + \sum_{n=1}^{M_k} q_n$$

satisfying

$$0 < S_k - r < p_{M_k}, \quad \text{and} \quad 0 < r - T_k < -q_{M_k}.$$

vi. Explain why the rearrangement

$$(b_n) = (p_1, \ldots, p_{N_1}, q_1, \ldots, q_{M_1}, p_{N_1+1}, \ldots, p_{N_2}, q_{M_1+1}, \ldots, q_{M_2}, \ldots),$$

satisfies $\sum b_n = r$. Deduce Riemann's Rearrangement Theorem.