PROBLEM SET 2: SOLUTIONS

A1) a) By Solution to September 11 Worksheet we have

$$\text{area of } K(m) = \text{area of } K(m-1) + \begin{pmatrix}\text{no. of}\\ \text{edges}\\ \text{of } K(m-1)\end{pmatrix} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3^2}\right)^m$$

$$= \underset{\displaystyle \text{area of } K(m-2) + \begin{pmatrix}\text{no. of edges}\\ \text{of } K(m-2)\end{pmatrix} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^{m-1}}{\underset{\Vert}{\text{area of } K(m-1)}} + 3 \cdot 4^{m-1} \cdot \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^m$$

$$= \text{area of } K(m-2) + \begin{pmatrix}\text{no. of edges}\\ \text{of } K(m-2)\end{pmatrix} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^{m-1} + 3 \cdot 4^{m-1} \cdot \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^m$$

$$= \text{area of } K(m-2) + 3 \cdot 4^{m-2} \cdot \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^{m-1} + 3 \cdot 4^{m-1} \cdot \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^m$$

$$= \text{area of } K(m-3) + 3 \cdot 4^{m-3} \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^{m-2} + 3 \cdot 4^{m-2} \cdot \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^{m-1} + 3 \cdot 4^{m-1} \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^m$$

$$\vdots$$

$$= \text{area of } K(0) + 3 \cdot 4^0 \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^1 + 3 \cdot 4^1 \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^2 + \dots + 3 \cdot 4^{m-1} \frac{\sqrt{3}}{4} \left(\frac{1}{9}\right)^m$$

$$= \frac{\sqrt{3}}{4} + \frac{3 \cdot \sqrt{3}}{4} \left(\frac{4}{4}\right) \cdot \left(\frac{1}{9}\right)^1 + \frac{3\sqrt{3}}{4} \left(\frac{4}{4}\right) 4^1 \left(\frac{1}{9}\right)^2 + \dots + \frac{3\sqrt{3}}{4} \left(\frac{4}{4}\right) 4^{m-1} \left(\frac{1}{9}\right)^m$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \cdot \left(\frac{4}{9}\right) + \frac{3\sqrt{3}}{16} \cdot \left(\frac{4}{9}\right)^2 + \dots + \frac{3\sqrt{3}}{16} \left(\frac{4}{9}\right)^m$$

$$= \boxed{\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \cdot \sum_{i=1}^{m} \left(\frac{4}{9}\right)^i}$$

b) $$\text{Area of } K(\infty) = \lim_{m\to\infty} (\text{area of } K(m))$$
$$= \lim_{m\to\infty} \left(\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \cdot \sum_{i=1}^{m} \left(\frac{4}{9}\right)^i \right)$$
$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \cdot \lim_{m\to\infty} \sum_{i=1}^{m} \left(\frac{4}{9}\right)^i$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \sum_{i=1}^{\infty} \left(\frac{4}{9}\right)^i \quad , \quad b|c \quad \sum_{i=1}^{\infty} \left(\frac{4}{9}\right)^i \overset{\text{def}}{=} \lim_{m \to \infty} \sum_{i=1}^{m} \left(\frac{4}{9}\right)^i$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \cdot \left(\frac{4}{9}\right) \frac{1}{\left(1 - \frac{4}{9}\right)} ,$$

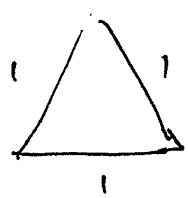
by Geometric Series Theorem

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16} \cdot \frac{4}{5}$$

$$= \frac{2\sqrt{3}}{5}$$

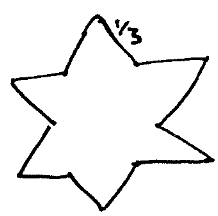
c) Let's investigate:

$m = 0$



perimeter $= 3$

$m = 1$



perimeter $= 12 \cdot \frac{1}{3}$

Guess: $\text{perimeter of } K(m) = \left(\text{no. of edges of } K(m)\right) \cdot \left(\text{length of edge}\right)$

$$= 3 \cdot 4^m \cdot \left(\frac{1}{3}\right)^m = 3 \cdot \left(\frac{4}{3}\right)^m$$

Since $\left(\frac{4}{3}\right)^m$ is unbounded as $m \to \infty$, we have

$$\text{perimeter of } K(\infty) = \lim_{m \to \infty} \left(\text{perimeter of } K(m)\right) = +\infty .$$

A2a) LCT: $a_n = \dfrac{n+5}{2n^3+1}$, $b_n = \dfrac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{n+5}{2n^3+1} \cdot \frac{n^2}{1} = \frac{n^3 + 5n^2}{2n^3+1} \xrightarrow{n \to \infty} \frac{1}{2} > 0$$

Hence, as $\sum b_n$ <u>convergent</u> the same is true of $\sum a_n$

$\boxed{\text{CONVERGENT}}$

(p-series $p = 2$)

b) LCT: $a_n = \frac{n+1}{n\sqrt{n}}$ $\quad b_n = \frac{1}{\sqrt{n}}$

$$\frac{a_n}{b_n} = \frac{n+1}{n\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \frac{n+1}{n} = 1 + \frac{1}{n} \longrightarrow 1 > 0$$

As $\sum b_n$ divergent (p-series, $p = 1/2$) same is true of $\sum a_n$.

$\boxed{\text{DIVERGENT}}$

c) LCT: $a_i = \frac{6^i}{5^i - 1} > 0$ $\quad b_i = \left(\frac{6}{5}\right)^i > 0$

$$\frac{a_i}{b_i} = \frac{6^i}{5^i - 1} \cdot \frac{5^i}{6^i} = \frac{5^i}{5^i - 1} = \frac{1}{1 - \left(\frac{1}{5}\right)^i} \xrightarrow{i \to \infty} 1 > 0$$

As $\sum b_i$ divergent (geom. series $r = 6/5 > 1$) same is true of $\sum a_i$

$\boxed{\text{DIVERGENT}}$

d) LCT: $a_n = \frac{\sqrt{n}}{n-1} > 0$, $\quad b_n = \frac{1}{\sqrt{n}}$

$$\frac{a_n}{b_n} = \frac{\sqrt{n}}{n-1} \cdot \frac{\sqrt{n}}{1} = \frac{n}{n-1} = \frac{1}{1 - \frac{1}{n}} \longrightarrow 1 > 0$$

As $\sum b_n$ divergent (p-series, $p = 1/2$) same is true of $\sum a_n$.

$\boxed{\text{DIVERGENT}}$

e) DCT: $a_n = \frac{1}{2n+3}$ $\quad b_n = \frac{1}{5n}$

For $n = 1, 2, 3, \ldots$ $\quad 5n \geq 2n + 3 \Rightarrow \frac{1}{2n+3} \geq \frac{1}{5n}$

As $\sum b_n$ divergent and $a_n \geq b_n$ for all n, $\sum a_n$ divergent

$\sum \frac{1}{5n} = \frac{1}{5} \sum \frac{1}{n}$ divergent.

$\boxed{\text{DIVERGENT}}$

f) DCT: $a_n = \frac{1}{3^n + 4^n}$ $\quad b_n = \frac{1}{4^n}$

For $n = 1, 2, \ldots$ $\quad 3^n + 4^n > 4^n \Rightarrow \frac{1}{4^n} > \frac{1}{3^n + 4^n}$

As $\sum b_n$ convergent (geom. series, $r = \frac{1}{4}$), the series $\sum a_n$ is convergent

$\boxed{\text{CONVERGENT}}$

g) $\sum_{m=1}^{\infty} \frac{1-(-1)^m}{m^4}$: $a_1 = \frac{2}{1}$ $a_2 = 0$

$a_3 = \frac{2}{3^4}$ $a_4 = 0$

$a_5 = \frac{2}{5^4}$ $a_6 = 0$

For each m, $a_m \leq \frac{2}{m^4}$.

Hence, by DCT with convergent series $\sum_{m=1}^{\infty} \frac{2}{m^4}$ (p-series, $p=4$), the series $\sum a_m$ convergent.

$\boxed{\text{CONVERGENT}}$

h) DCT: Let $a_n = \frac{n!}{n^n}$. Then

$$0 < a_n = \frac{1}{n}\cdot\frac{2}{n}\cdot\left(\frac{3}{n}\right)\left(\frac{4}{n}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)$$

$$\leq \frac{1}{n}\cdot\frac{2}{n}\,1\cdot 1\cdots\cdots 1\cdot 1 = \frac{1}{n}\cdot\frac{2}{n} = \frac{2}{n^2}$$

By DCT with $\sum \frac{2}{n^2}$, convergent series (p-series, $p=2$), the series $\sum a_n$ is convergent

$\boxed{\text{CONVERGENT}}$

i) $\sum_{k=1}^{\infty} \frac{1}{2^k k!}$: For each $k = 1, 2, 3, \ldots$

$$0 < \frac{1}{2^k k!} \leq \frac{1}{2^k}$$

By DCT with convergent series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ (geom. series with $r = \frac{1}{2}$), the series $\sum \frac{1}{2^k k!}$ is convergent

$\boxed{\text{CONVERGENT}}$

j) Let $a_j = \frac{1+j!}{(1+j)!}$

$$= \frac{1+j!}{1.2.3.\cdots(j-1)\cdot j\cdot(j+1)}$$

$$= \frac{1}{(j+1)!} + \frac{1.2.3\cdots j}{1.2.3\cdots j\cdot(j+1)} = \frac{1}{(j+1)!} + \frac{1}{j+1}$$

As j gets very large, a_j "looks like" $\frac{1}{j+1}$

Try: LCT with $b_j = \frac{1}{j+1}$

$$\frac{a_j}{b_j} = \frac{1+j!}{(1+j)!} \cdot \frac{(j+1)}{1} = \frac{1+j!}{j!} = \frac{1}{j!} + 1 \longrightarrow 1 > 0$$

Hence, ~~by~~ as $\sum \frac{1}{j+1}$ is divergent the same is true of $\sum a_j$.

$\boxed{\text{DIVERGENT}}$

A3) a) $x = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots$

~~b)~~ i), ii)
$$= 9 \cdot \left(\sum_{i=1}^{\infty} \frac{1}{10^i} \right)$$

$$\underset{(r=\frac{1}{10})}{\overset{\text{Geom. series Theorem}}{=}} 9 \cdot \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{9}{9} = 1.$$

iii) ~~c)~~ at least 2

iv) ~~d)~~ rational numbers admitting at least one terminating decimal representation.

b) i) $0.888\ldots = \frac{8}{10} + \frac{8}{10^2} + \dots$

$$= 8 \cdot \left(\frac{1}{10} + \frac{1}{10^2} + \dots \right)$$

$$= 8 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i} = 8 \cdot \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}}$$

$$= \boxed{\frac{8}{9}}$$

ii) $0.121212\ldots$

$$= \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \dots$$

$$= \frac{1}{10} + \frac{1}{10^3} + \frac{1}{10^5} + \dots$$

$$+ \frac{2}{10^2} + \frac{2}{10^4} + \frac{2}{10^6} + \dots$$

$$= \cancel{\tfrac{4}{12}}\ \frac{10}{10} \cdot \left(\frac{1}{10} + \frac{1}{10^3} + \frac{1}{10^5} + \cdots \right)$$

$$+ \quad 2 \cdot \left(\sum_{i=1}^{\infty} \left(\frac{1}{10^2} \right)^i \right)$$

$$= 10 \cdot \sum_{i=1}^{\infty} \frac{1}{10^{2i}} \quad + \quad 2 \cdot \left(\sum_{i=1}^{\infty} \frac{1}{10^{2i}} \right)$$

$$= 10 \cdot \frac{1}{10^2} \left(\frac{1}{1 - \frac{1}{10^2}} \right) + 2 \cdot \frac{1}{10^2} \cdot \frac{1}{1 - \frac{1}{10^2}}$$

$$= \frac{10 + 2}{99} = \frac{12}{99}.$$

iii) $0.3464646 = \frac{3}{10} + \frac{4}{10^2} + \frac{6}{10^3} + \frac{4}{10^4} + \frac{6}{10^5} + \cdots$

$$= \frac{3}{10} + 4 \left(\sum_{i=1}^{\infty} \left(\frac{1}{10^2} \right)^i \right) + \frac{6}{10} \sum_{i=1}^{\infty} \left(\frac{1}{10^2} \right)^i$$

$$= \frac{3}{10} + 4 \cdot \frac{1}{10^2} \cdot \frac{1}{1 - \frac{1}{10^2}} + \frac{6}{10} \cdot \frac{1}{10^2} \cdot \frac{1}{1 - \frac{1}{10^2}}$$

$$= \frac{3}{10} + \frac{4}{99} + \frac{6}{990}$$

$$= \frac{297 + 40 + 6}{990} = \frac{343}{990}$$

iv) $2.53165165\ldots$

$$= 2 + \frac{53}{100} + \frac{1}{10^3} + \frac{6}{10^4} + \frac{5}{10^5} + \frac{1}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

$$= 2 + \frac{53}{100} + \left(\frac{1}{10^3} + \frac{1}{10^6} + \cdots \right)$$

$$+ \frac{6}{10} \left(\frac{1}{10^3} + \frac{1}{10^6} + \cdots \right)$$

$$+ \frac{5}{10^2} \left(\frac{1}{10^3} + \frac{1}{10^6} + \cdots \right)$$

$$= 2 + \frac{53}{100} + \frac{1}{10^3} \cdot \frac{1}{1 - \frac{1}{10^3}} + \frac{6}{10} \cdot \frac{1}{10^3} \cdot \frac{1}{1 - \frac{1}{10^3}}$$

$$+ \frac{5}{10^2} \cdot \frac{1}{10^3} \cdot \frac{1}{1 - \frac{1}{10^3}}$$

$$= 2 + \frac{53}{100} + \frac{1}{999} + \frac{6}{9990} + \frac{5}{99900}$$

$$= 2 + \frac{53}{100} + \frac{100 + 60 + 5}{99900}$$

$$= 2 + \frac{53}{100} + \frac{165}{99900} = \boxed{\frac{21076}{8325}}$$

c) $$0.ababab\ldots = .a\left(\frac{1}{10} + \frac{1}{10^3} + \cdots\right) + b.\left(\frac{1}{10^2} + \frac{1}{10^4} + \cdots\right)$$

$$= 10a\left(\frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) + b\left(\frac{1}{10^2} + \frac{1}{10^4} + \cdots\right)$$

$$= (10a + b) \cdot \frac{1}{10^2} \cdot \frac{1}{1 - \frac{1}{10^2}} = \frac{10a + b}{99}$$

ii) Take $a = 2$, $b = 4$.

(A4) a) $a_n = \frac{(-1)^n}{n^2+1}$, $|a_n| = \frac{1}{n^2+1}$

Since $\sum \frac{1}{n^2+1}$ is convergent (use DCT with $\sum \frac{1}{n^2}$)

the series $\sum a_n$ is abs. convergent.

b) $a_n = \frac{(-2)^n}{n!}$, $|a_n| = \frac{2^n}{n!}$

we have $$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2}{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}$$

$$\leq \frac{2}{1} \cdot 1 \cdot 1 \cdots \frac{2}{(n-1)} \cdot \frac{2}{n} = \frac{8}{n(n-1)}$$

The series $\sum_{n=2}^{\infty} \frac{8}{n(n-1)}$ is convergent (telescoping series; or use LCT with $\sum_{n=2}^{\infty} \frac{1}{n^2}$), hence by DCT

$\sum \frac{2^n}{n!}$ is convergent. $\Rightarrow$ $\sum a_n$ abs. convergent

c) $\sum_{n=4}^{\infty} \frac{n!}{(-100)^n}$: for $n \geq 200$

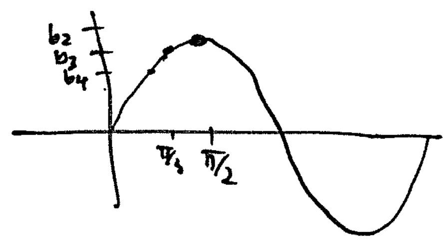
$$\frac{n!}{(-100)^n} = (-1)^n \frac{1\cdot 2 \cdots 100 \cdot 101 \cdot 102 \cdots 200 \cdots n}{1\cdot 1 \cdots 1 \cdot 100 \cdot 100 \cdots 100 \cdot 1 \cdots 1}$$

$$\begin{cases} \geq 100! \cdot 1 \cdot 1 \cdots 1 \cdot n & , n \text{ even} \\ \leq -100! \cdot 1 \cdot 1 \cdots 1 \cdot n & , n \text{ odd} \end{cases}$$

Hence, a_n unbounded $\Rightarrow \lim a_n \neq 0$
$\Rightarrow \sum a_n$ divergent.

d) $\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{n}$. Let $b_n = \sin(\pi/n)$.

$b_1 = \sin(\pi)$
$b_2 = \sin(\pi/2)$
$b_3 = \sin(\frac{\pi}{3})$ ...



For $n \geq 2$, $(b_n)_{n\geq 2}$ is decreasing and $\lim b_n = 0$.
Hence; by AST, $\sum (-1)^n b_n$ is convergent.

Consider $\sum_{n=1}^{\infty} \sin \pi/n$. Use $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. (L'Hopital Rule)

We have $\lim_{n\to\infty} \frac{\sin(\pi/n)}{(\pi/n)} = 1$.

Hence, by LCT with divergent series $\sum_{n=1}^{\infty} \pi/n$ the series $\sum \sin \pi/n$ is divergent.

$\boxed{\text{CONDITIONALLY CONVERGENT}}$

e) ABS. CONV: $\sum_{n=2}^{\infty} \left(\frac{5}{64}\right)^n$ is convergent by Geom. Series Theorem.

f) ABS CONV: We have $\frac{n!}{n^n} = \frac{1\cdot 2\cdot 3 \cdots (n-1)\cdot n}{n\cdot n\cdot n \cdots n\cdot n}$

$\leq \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots 1 \cdot 1 = \frac{2}{n^2}$

Hence, by DCT with $\sum \frac{2}{n^2}$ the series $\sum \frac{n!}{n^n}$ converges. Hence, $\sum (-1)^n \frac{n!}{n^n}$ is abs. conv.

A5 a) Let $b_n = \frac{1}{n!}$ Then,

$$b_n = \frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdots n} \leq \frac{1}{(n-1) \cdot n}$$

The series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ is convergent (telescoping series)

so, by DCT, $\sum b_n$ is convergent.

Hence, $\sum (-1)^n \frac{1}{n!}$ is abs. convergent $\Rightarrow$ convergent.