Middlebury College

# Calculus II: Fall 2017
## Problem Set 1 *Solutions*
**Contact:** gmelvin@middlebury.edu

## Problems for submission

A1. Consider the sequence $(a_n)$, where $a_n = \frac{1}{n^3+10}$. In this problem you will show that $(a_n)$ is convergent with limit $L = 0$ using the Squeeze Theorem.

(a) Carefully explain why the sequence $(b_n)$, where $b_n = 0$, is convergent with limit $L = 0$.

(b) Carefully explain why the sequence $(c_n)$, where $c_n = \frac{1}{n^3}$, is convergent with limit $L = 0$.

(c) Using the Squeeze Theorem, carefully explain why $(a_n)$ is convergent with limit $L = 0$.

**Solution:**

(a) Let $L = 0$. Since $b_n = 0$, for every $n$ we find that, given any $\varepsilon > 0$,

$$n \geq 1 \quad \Longrightarrow \quad |b_n - 0| = |0| = 0 < \varepsilon.$$

This shows that $(b_n)$ is convergent with limit $L = 0$.

(b) Let $c_n = \frac{1}{n^3}$. Using the Limit Laws for Sequences we have

$$\lim_{n\to\infty} c_n = \left(\lim_{n\to\infty} \frac{1}{n}\right)^3 = 0^3 = 0,$$

where we have used that $\lim_{n\to\infty} = 0$.

(c) First, we observe that $a_n > 0$, for each $n = 1, 2, 3, \ldots$. Also, for each $n = 1, 2, 3, \ldots$,

$$n^3 + 10 > n^3 \quad \Longrightarrow \quad \frac{1}{n^3+10} < \frac{1}{n^3}.$$

Hence, by the Squeeze Theorem, we conclude that $\lim_{n\to\infty} a_n = 0$.

A2. We introduce the following definitions:

- a sequence $(a_n)$ is **bounded above** if there exists $M$ such that $a_n \leq M$, for every $n \geq 1$. We call $M$ an **upper bound** of $(a_n)$.
- a sequence $(a_n)$ is **bounded below** if there exists $m$ such that $a_n \geq m$, for every $n \geq 1$. We call $m$ a **lower bound** of $(a_n)$.
- a sequence $(a_n)$ is **alternating** if $a_n a_{n+1} < 0$, for every $n \geq 1$; that is, any two consecutive terms must have opposite sign.

For each of the following sequences $(a_n)$[1], determine which of the properties hold. You do not have to provide justification.

(i) bounded above, bounded below. If bounded above (resp. below) provide an explicit upper (resp. lower) bound.

(ii) increasing, decreasing or alternating.

(iii) convergent, divergent.

[1] Given a natural number $n$, define $n!$ (*n factorial*) to be the product $n! \overset{def}{=} 1 \cdot 2 \cdot 3 \cdots (n-2) \cdots (n-1) \cdot n$.

a) $a_n = \frac{2n^2}{n^2+1}$ b) $a_n = \frac{2n}{n^2+1}$ c) $a_n = \sin(1/n)$ d) $a_n = 4 - \frac{(-1)^n}{n}$ e) $a_n = \frac{n^2-(-1)^n}{n}$
f) $a_n = \frac{2^n}{(-\pi)^n}$ g) $a_n = \frac{5-2n}{n+5}$ h) $a_n = \frac{\sin(n)}{n}$ i) $a_n = \sqrt{n+1} - \sqrt{n}$ j) $a_n = \frac{(n!)^2}{(2n)!}$.

**Solution:** We write BA for 'bounded above', BB for 'bounded below', I for 'increasing', D for 'decreasing', A for 'alternating', C for 'convergent', V for 'divergent'.

a) BA ($M = 2$), BB ($m = 0$), I, C b) BA ($M = 10$), BB ($m = 0$), D, C
c) BA ($M = 1$), BB ($m = 0$), D, C d) BA ($M = 5$), BB ($m = 3$), C
e) BB ($m = 0$), D f) BA ($M = 1$), BB ($m = -1$), A, C
g) BA ($M = 5$), BB ($m = -2$), C h) BA ($M = 1$), BB ($m = -1$), C
i) BA ($M = 1$), BB ($m = 0$), D, C j) BA ($M = 1$), BB ($m = 0$), D, C.

A3. Consider the sequence $(a_n)$, where $a_n = \frac{1}{n^2+1}$. In this problem you will show *directly* that $(a_n)$ is convergent with limit $L = 0$.

(a) Explain why $|a_n| = a_n$, for every natural number $n$.
(b) Determine a natural number $N$ so that, if $n \geq N$ then $|a_n| < 20$.
(c) Determine a natural number $N$ so that, if $n \geq N$ then $|a_n| < \frac{1}{10}$.
(d) Determine a natural number $N$ so that, if $n \geq N$ then $|a_n| < 2^{-16}$.
(e) Let $\varepsilon > 0$. Determine a natural number $N$ so that, if $n \geq N$ then $|a_n| < \varepsilon$. (*Hint: the natural number $N$ will depend on $\varepsilon$.*)

**Solution:**

(a) Since $\frac{1}{n^2+1} > 0$, for every $n = 1, 2, 3, \ldots$, we have $|a_n| = a_n$.

(b) We have

$$|a_n| < 20 \quad \Leftrightarrow \quad \frac{1}{n^2+1} < 20 \quad \Leftrightarrow \quad n^2 > \frac{-19}{20}$$

This last inequality is true for every natural number $n$. Hence, we can take $N = 1$. Then, for each natural number $n \geq N = 1$ we have

$$n \geq N = 1 \quad \Longrightarrow \quad n^2 \geq N^2 = 1 > -\frac{19}{20} \quad \Longrightarrow \quad |a_n| < 20.$$

(c) In a similar way, we see that

$$|a_n| < \frac{1}{10} \quad \Leftrightarrow \quad \frac{1}{n^2+1} < \frac{1}{10} \quad \Leftrightarrow \quad n^2 > 9$$

This last inequality is true for every natural number $n \geq 4$. Hence, we can take $N = 4$. Then, for each natural number $n \geq N = 4$ we have

$$n \geq N = 4 \quad \Longrightarrow \quad n^2 \geq N^2 = 16 > 9 \quad \Longrightarrow \quad |a_n| < \frac{1}{10}.$$

(d) In a similar way, we see that

$$|a_n| < 2^{-16} \quad \Leftrightarrow \quad \frac{1}{n^2+1} < 2^{-16} \quad \Leftrightarrow \quad n^2 > 2^{16} - 1 = 65535$$

This last inequality is true for every natural number $n \geq 1000$ (for example). Hence, we can take $N = 1000$. Then, for each natural number $n \geq N = 1000$ we have

$$n \geq N = 1000 \quad \Longrightarrow \quad n^2 \geq N^2 = 1,000,000 > 65535 \quad \Longrightarrow \quad |a_n| < 2^{-16}.$$

(e) Let $\varepsilon > 0$. Then, we see that

$$|a_n| < \varepsilon \quad \Leftrightarrow \quad \frac{1}{n^2+1} < \varepsilon \quad \Leftrightarrow \quad n^2 > \frac{1}{\varepsilon} - 1$$

We want to know when this last inequality is true. First we see that, if $\varepsilon \geq 1$ then this last inequality is true for every natural number $n$ (since the right hand side is $\leq 0$). Hence, if we take $N = 1$ whenever $\varepsilon \geq 1$ then

$$n \geq N = 1 \quad \Longrightarrow \quad n^2 \geq N^2 = 1 > \frac{1}{\varepsilon} - 1 \quad \Longrightarrow \quad |a_n| < \varepsilon.$$

Suppose that $0 < \varepsilon < 1$: hence, $\frac{1}{\varepsilon} - 1 > 0$. Let $N$ be a natural number satisfying $N > \sqrt{\frac{1}{\varepsilon} - 1}$. Then,

$$n \geq N > \sqrt{\frac{1}{\varepsilon} - 1} \quad \Longrightarrow \quad n^2 \geq N^2 > \frac{1}{\varepsilon} - 1 \quad \Longrightarrow \quad |a_n| < \varepsilon.$$

A4. Consider the series $\sum_{n=1}^{\infty}(-1)^n$.

(a) Write down the first five partial sums $s_1, s_2, s_3, s_4, s_5$. What is a general expression for $s_n$?

(b) Is the series $\sum_{n=1}^{\infty}(-1)^n$ convergent or divergent? Explain your answer carefully.

**Solution:**

(a)

$$s_1 = -1, \quad s_2 = (-1) + 1 = 0, \quad s_3 = (-1) + 1 + (-1) = -1,$$
$$s_4 = (-1) + 1 + (-1) + 1 = 0, \quad s_5 = (-1) + 1 + (-1) + 1 + (-1) = -1.$$

The general expression is

$$s_n = \begin{cases} -1, & \text{when } n \text{ odd}, \\ 0, & \text{when } n \text{ even}. \end{cases}$$

(b) This series is divergent: the sequences of partial sums does not converge to any fixed limit $L$.

A5. For each of the following series determine whether the series converges or diverges; make sure you justify your conclusion. If the series converges determine its limit.

a) $\sum_{n=1}^{\infty} \frac{1}{5^n}$ b) $\sum_{n=1}^{\infty} 3\left(\frac{-1}{4}\right)^{n-1}$ c) $\sum_{n=0}^{\infty} \frac{5}{10^{3n}}$ d) $\sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}}$ e) $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}}$

f) $\sum_{n=1}^{\infty} \frac{3+2^n}{2^{n+2}}$ g) $\sum_{n=1}^{\infty} \frac{3+2^n}{3^{n+2}}$ h) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ i) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ j) $\sum_{n=1}^{\infty} \frac{n}{2n+1}$.

**Solution:**

(a) Geometric series; converges to $1/4$.

(b) Geometric series; converges to $-3/5$.

(c) Geometric series; converges to $5000/999$.

(d) Geometric series; converges to $1/(2+\pi)^8((2+\pi)^2 - 1)$.

(e) Geometric series; converges to $64/345$.

(f) Divergent; the series can be split into a sum of two series $\sum_{n=1} \frac{3}{2^{n+2}} + \sum_{n=1}^{\infty} \frac{1}{4}$. The latter series does not converge: its terms do not converge to 0 as $n \to \infty$.

(g) Sum of two geometric series; converges to 7/18.

(h) This is a telescoping series: we can write

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

Then, the $n^{th}$ partial sum is

$$s_n = \frac{1}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{n} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

The sequence $(s_n)$ converges to 3/4. Hence, the series is convergent with limit 3/4.

(i) Observe that

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Thus, the $n^{th}$ partial sum is

$$s_n = \frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} - \frac{1}{3} - \frac{1}{5} - \ldots - \frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

The sequence $(s_n)$ converges to 1/2. Hence, the series is convergent with limit 1/2

(j) Divergent; the sequence of terms $\left(\frac{n}{2n+1}\right)_{n\geq 1}$ of the series do not converge to 0.

**Additional recommended problems (not for submission)**

B1. Let $f(x)$ be a differentiable function, defined for all $1 \leq x < \infty$ (and possibly on a larger domain). Consider the sequence $(a_n)_{n\geq 1}$, where $a_n = f(n)$.

(a) Suppose that $f'(x) \geq 0$, for all $1 \leq x < \infty$. Is the sequence $(a_n)$ increasing, decreasing or neither?

(b) Suppose that $f'(x) \leq 0$, for all $1 \leq x < \infty$. Is the sequence $(a_n)$ increasing, decreasing or neither?

(c) Consider the sequence $(a_n)$, where $a_n = \cos\left(\frac{\pi}{n}\right)$. In this problem you will use your results above to show that $(a_n)$ is a convergent sequence.

i. Let $f(x) = \cos\left(\frac{\pi}{x}\right)$. Show that $f'(x) = \frac{\pi \sin(\pi/x)}{x^2}$. (*Hint: chain rule!*)

ii. Explain why $\sin(\pi/x) \geq 0$ whenever $x \geq 1$.

iii. Explain carefully why $(a_n)$ is an increasing sequence.

iv. Show that $(a_n)$ is a bounded sequence and deduce that $(a_n)$ is a convergent sequence.

v. What do you think the limit of the sequence $(a_n)$ is? Justify your answer.

**Solution:**

(a) Since $f'(x) \geq 0$, the function $f(x)$ is increasing. Hence, the sequence $(a_n)$ is also increasing.

(b) Since $f'(x) \leq 0$, the function $f(x)$ is decreasing. Hence, the sequence $(a_n)$ is also decreasing.

(c) i. Using the chaine rule, we have

$$\begin{aligned} f'(x) &= (-\sin\left(\frac{\pi}{x}\right)) \cdot \left(-\frac{\pi}{x^2}\right) \\ &= \frac{\pi \sin(\pi/x)}{x^2}. \end{aligned}$$

ii. If $x \geq 1$ then $0 < \pi/x \leq \pi$. Hence, $\sin(\pi/x) \geq 0$.

iii. Since $f'(x) \geq 0$ whenever $x \geq 1$ (using the previous problem), we can deduce that $a_n$ is an increasing sequence, by (a).

iv. Since $-1 \leq f(x) \leq 1$, for all $x$, we conclude that $-1 \leq a_n \leq 1$, for each $n$. Hence, $(a_n)$ is an increasing, bounded sequence and therefore convergent, by M+B Theorem.

v. As $n \to \infty$, the quantity $\pi/n \to 0$. Hence, we expect that $\cos(\pi/n) \to \cos(0) = 1$: this is justified because $\cos(x)$ is a continuous function.

B2. Let $(a_n)$ be a sequence satisfying $a_n > 0$, for every $n = 1, 2, 3, \ldots$. Complete the following statements:

(a) $(a_n)$ is increasing is equivalent to $\frac{a_{n+1}}{a_n} \geq \underline{1}$, for every $n = 1, 2, 3, \ldots$.

(b) $(a_n)$ is decreasing is equivalent to $\frac{a_{n+1}}{a_n} \leq \underline{1}$, for every $n = 1, 2, 3, \ldots$.

B3. Let $(a_n)$ be a sequence.

- We say that $(a_n)$ **diverges to** $+\infty$ if, for every real number $K$ there exists a natural number $N$ such that
$$n \geq N \implies a_n > K.$$
We write, by abuse of notation, $\lim_{n\to\infty} a_n = +\infty$.

- We say that $(a_n)$ **diverges to** $-\infty$ if, for every real number $K$ there exists a natural number $N$ such that
$$n \geq N \implies a_n < K.$$
We write, by abuse of notation, $\lim_{n\to\infty} a_n = -\infty$.

(a) Determine whether the given sequence $(a_n)$ diverges to $+\infty$, $-\infty$, neither.

a) $a_n = n$ b) $a_n = (-1)^n n$ c) $a_n = 2n + (-1)^n$ d) $\frac{n^2-4}{n+5}$ e) $a_n = \frac{(2n)!}{2(n!)}$

(b) Let $c$ be a real number. Give an example of sequences $(a_n)$, $(b_n)$ such that $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = -\infty$ and $\lim_{n\to\infty}(a_n + b_n) = c$.

(c) Suppose that $(a_n)$ is a sequence that diverges to $+\infty$.

i. Let $(b_n)$ be a sequence such that $b_n \geq a_n$, for each $n = 1, 2, 3, \ldots$. Does $(b_n)$ diverge to $+\infty$, $-\infty$,neither $\pm\infty$, or is there not enough information to decide?

ii. Let $(b_n)$ be a sequence such that $b_n \leq a_n$, for each $n = 1, 2, 3, \ldots$. Does $(b_n)$ diverge to $+\infty$, $-\infty$,neither $\pm\infty$, or is there not enough information to decide?

iii. Let $(b_n)$ be a sequence such that $b_{2n} \geq a_{2n}$, for each $n = 1, 2, 3, \ldots$. Does $(b_n)$ diverge to $+\infty$, $-\infty$,neither $\pm\infty$, or is there not enough information to decide?

**Solution:**

(a) a) $+\infty$: given $K$, let $N$ be a natural number satisfying $N > K$. Then, for any $n \geq N$ we have $a_n = n \geq N > K$.

b) neither: the sequence oscillates between positive and negative terms.

c) Write down the first few terms of the sequence: $1, 5, 5, 9, 9, 13, 13, \ldots$. We see that $a_n$ diverges to $+\infty$.

d) As $n$ gets very large the terms $a_n$ will 'look like' $\frac{n^2}{n} = n$. Hence, we expect that for $n$ very large the sequence behaves like the sequence $b_n = n$: so, $(a_n)$ diverges to $+\infty$. Let's show this: we use the following trick

$$a_n \frac{n^2-4}{n+5} = \frac{n^2 - 25 + 25 - 4}{n+5} = \frac{(n-5)(n+5) + 21}{n+5} = n - 5 + \frac{21}{n+5}$$

Hence, for each $n = 1, 2, 3, \ldots$, we have $a_n \geq n - 5$ because $\frac{21}{n+5} \geq 0$. So, given a real number $K$, take a natural number $N$ such that $N - 5 > K$. Then,

$$n \geq N \quad \Longrightarrow \quad a_n \geq n - 5 \geq N - 5 > K.$$

e) We see that

$$\frac{(2n)!}{2(n!)} = \frac{2n(2n-1)\cdots(n+1)n(n-1)\cdots 2.1}{2.n(n-1)\cdots 2.1} = \frac{2n(2n-1)\cdots(n+1)}{2}$$

For each $n = 1, 2, 3, \ldots$, $\frac{n+1}{2} \geq 1$, so we see that

$$a_n = \frac{(2n)!}{2(n!)} \geq 2n(2n-1)\cdots(n+2) \geq n + 2$$

Hence, as $n$ gets very large, the terms $a_n$ get pushed to $+\infty$: this means that $(a_n)$ diverges to $+\infty$.

To show this, suppose given a real number $K$. Then, take $N$ a natural number such that $N + 2 > K$. Hence,

$$n \geq N \quad \Longrightarrow \quad a_n \geq n + 2 \geq N + 2 > K.$$

We have shown that $(a_n)$ diverges to $+\infty$.

(b) Take $a_n = n$, $b_n = -n$ and $c = 0$. Then, $(a_n)$ diverges to $+\infty$, $(b_n)$ diverges to $-\infty$ and $(a_n + b_n)$ is the constant sequence $(0)$ (i.e. all the terms are 0).

(c) i. $(b_n)$ diverges to $+\infty$

ii. there is not enough information to decide

iii. there is not enough information to decide

B4. Consider the sequence $(a_n)$, where

$$a_n = \frac{\alpha_r n^r + \alpha_{r-1} n^{r-1} + \ldots + \alpha_1 n + \alpha_0}{\beta_s n^s + \beta_{s-1} n^{s-1} + \ldots \beta_1 n + \beta_0}.$$

Here $\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_s$ are constants, $\alpha_r, \beta_s \neq 0$.

(a) Let $r < s$. Use the Limit Laws for Sequences to show that $(a_n)$ is convergent with limit $L = 0$.

(b) Let $r = s$. Use the Limit Laws for Sequences to show that $(a_n)$ is convergent with limit $L = \frac{\alpha_r}{\beta_s}$.

**Solution:** We have

$$a_n = \frac{\alpha_r n^r + \alpha_{r-1} n^{r-1} + \ldots + \alpha_1 n + \alpha_0}{\beta_s n^s + \beta_{s-1} n^{s-1} + \ldots \beta_1 n + \beta_0} = \frac{n^r}{n^s}\left(\frac{\alpha_r + \alpha_{r-1}/n + \ldots + \alpha_1/n^{r-1} + \alpha_0/n^r}{\beta_s + \beta_{s-1}/n^s + \ldots \beta_1/n^{s-1} + \beta_0/n^s}\right)$$

$$= n^{r-s}\left(\frac{\alpha_r + \alpha_{r-1}/n + \ldots + \alpha_1/n^{r-1} + \alpha_0/n^r}{\beta_s + \beta_{s-1}/n^s + \ldots \beta_1/n^{s-1} + \beta_0/n^s}\right)$$

(a) If $r < s$ then $r - s < 0$ and $\lim n^{r-s} = 0$. Hence,

$$\lim a_n = (\lim n^{r-s}) \cdot \frac{\alpha_r}{\beta_s} = 0$$

(b) Similarly, if $r = s$ then $\lim n^{r-s} = 1$ and $\lim a_n = \frac{\alpha_r}{\beta_s}$.

B5. (*For students who have seen mathematical induction*) In this problem you will determine an approach to approximating the real number $\sqrt{2}$. Let $a_1 = 1$, and define $a_{n+1} = \sqrt{1 + 2a_n}$, for $n = 1, 2, 3, \ldots$.

(a) Write down the first five terms $a_1, a_2, a_3, a_4, a_5$.

(b) Show that the sequence $(a_n)$ is increasing. (*Hint: use induction*)

(c) Show that the sequence $(a_n)$ is bounded above by 3. (*Hint: use induction*)

(d) Deduce that $(a_n)$ is convergent. Let $L = \lim_{n\to\infty} a_n$ denote the (yet to be determined) limit of $(a_n)$.

(e) Consider the sequence $(b_n)$, where $b_n = \sqrt{1+2a_n}$. Explain carefully why $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$. Deduce that $L = 1+\sqrt{2}$.

(f) Use the previous problem to describe an approach to determine an approximation of the real number $\sqrt{2}$ to within 10 decimal places.

**Solution:**

(a) $a_1 = 1, a_2 = \sqrt{3}, a_3 = \sqrt{1+2\sqrt{3}}, a_4 = \sqrt{1+2\sqrt{1+2\sqrt{3}}}, a_5 = \sqrt{1+2\sqrt{1+2\sqrt{1+2\sqrt{3}}}}$.

(b) Proceed by mathematical induction. We have $a_2 > a_1$, by inspection. Assume $a_k > a_{k-1}$. We will show that this implies $a_{k+1} > a_k$. Indeed, we have

$$a_{k+1} = \sqrt{1+2a_k} > \sqrt{1+2a_{k-1}} = a_k.$$

Hence, $a_{n+1} > a_n$, for all $n$ by induction.

(c) We proceed by mathematical induction. We have $a_1 = 1 < 3$. Assume that $a_k < 3$. We will show that this implies that $a_{k+1} < 3$. Indeed, we have

$$a_{k+1} = \sqrt{1+2a_k} < \sqrt{1+2.3} = \sqrt{7} < 3.$$

(d) The sequence $(a_n)$ is increasing and bounded. Hence, by the M+B Theorem, $(a_n)$ is convergent. Let $L = \lim_{n\to\infty} a_n$.

(e) Since $b_n = a_{n+1}$, we have $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_{n+1}$. Hence, $(b_n)$ is convergent also. Thus,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+2a_n} = \sqrt{1+2\lim_{n\to\infty}} = \sqrt{1+2L}.$$

Hence,

$$L^2 = 1+2L \quad\Longrightarrow\quad L^2 - 2L - 1 = 0.$$

$L$ must be a root of the equation $x^2 - 2x - 1 = 0$. Using the quadratic formula we know that the roots are

$$1 \pm \sqrt{2}.$$

Since $1-\sqrt{2} < 0$, and the sequence $(a_n)$ was increasing (hence, $a_n \geq a_1 = 1$), we must have that $L = 1+\sqrt{2}$.

(f) Since $\lim_{n\to\infty} a_n = 1+\sqrt{2}$, we could

B6. Is the following sequence convergent? If yes, determine its limit; if no, explain why not.

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

(a) The sequence is convergent. We proceed as in the previous exercise. First, note that

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}, \quad n = 1, 2, 3, \ldots$$

$(a_n)$ is increasing: proceed by induction, as above. We have $a_2 - a_1 > 0$ by inspection. Suppose that $a_k > a_{k-1}$. We will show that $a_{k+1} > a_k$. Indeed, we have

$$a_{k+1} = \sqrt{2a_k} > \sqrt{2a_{k-1}} = a_k.$$

Hence, $(a_k)$ is increasing, by induction.

$(a_n)$ is bounded: we claim that $a_n < 3$, for all $n$. First, we have $a_1 = \sqrt{2} < 3$. Assume that $a_k < 3$. We will show that $a_{k+1} < 3$. Indeed,

$$a_{k+1} = \sqrt{2a_k} < \sqrt{2.3} = \sqrt{6} < 3.$$

Hence, $a_n < 3$, for all $n$ by induction.

Thus, $(a_n)$ is an increasing, bounded sequence, therefore it is convergent. Let $L = \lim a_n$. Hence, we have

$$L = \lim a_{n+1} = \lim \sqrt{2a_n} = \sqrt{2 \lim a_n} = \sqrt{2L}.$$

Hence, $L^2 = 2L$. As $(a_n)$ is increasing, so $a_n \geq a_1 = \sqrt{2}$ we must have that $L = 2$. Hence, $(a_n)$ is convergent with limit 2.

B7. Let $(a_n)$ be a sequence.

- We say that $(a_n)$ is **eventually increasing** if there is some natural number $E$ such that the sequence $(a_n)_{n \geq E}$ is increasing.
- We say that $(a_n)$ is **eventually decreasing** if there is some natural number $F$ such that the sequence $(a_n)_{n \geq F}$ is decreasing.

Suppose that $(a_n)$ is eventually increasing/eventually decreasing and bounded. Show that $(a_n)$ is convergent.

B8. In this problem you will show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(a) Consider the series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ and denote its partial sums $s_1, s_2, s_3, \ldots$.

i. Write down the partial sums $s_2, s_3, s_4, s_5, s_6$.

ii. Show that $\sum_{n=2}^{m} \frac{1}{n(n-1)} = 1 - \frac{1}{m}$, for $m = 2, 3, 4, \ldots$.

iii. Conclude that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges and determine its limit.

(b) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and denote its partial sums $t_1, t_2, t_3, \ldots$. Show that $t_n \leq 1 + s_n$, for $n = 2, 3, 4, \ldots$ and deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (*Hint: show that the sequence* $(t_m)_{m \geq 1}$ *is increasing*)

**Solution:**

(a) i. $s_2 = \frac{1}{2} s_3 = \frac{2}{3}, s_4 = \frac{3}{4}, s_5 = \frac{4}{5}, s_6 = \frac{5}{6}$.

ii. We have

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

Hence,

$$s_m = \sum_{n=2}^{m} \frac{1}{n(n-1)} = \sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m-1}\right) - \left(\frac{1}{2} + \ldots + \frac{1}{m}\right) = 1 - \frac{1}{m}$$

iii. Thus the sequence of partial sums $(s_m)_{m \geq 2}$ is convergent with limit 1: use the above expression to see that $s_m = 1 - \frac{1}{m} \to 1$ as $m \to \infty$.

(b) The sequence $(t_m)$ is increasing since

$$t_{m+1} = \sum_{n=1}^{m+1} \frac{1}{n^2} = \sum_{n=1}^{m} \frac{1}{n^2} + \frac{1}{(m+1)^2} = t_m + \frac{1}{(m+1)^2} \geq t_m.$$

Moreover, for each $n = 1, 2, 3, \ldots$,

$$n^2 \geq n(n-1) \quad \Longrightarrow \quad \frac{1}{n^2} \leq \frac{1}{n(n-1)}$$

and

$$t_m = \sum_{n=1}^{m} \frac{1}{n^2} = 1 + \sum_{n=2}^{m} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{m} \frac{1}{n(n-1)} = 1 + s_m < 1 + 1 = 2.$$

The last inequality holds because $(s_m)$ is an increasing sequence with limit 1. Thus, $(t_m)$ is a bounded, increasing sequence, and hence convergent by the M+B Theorem.

### Challenging Problems

C1. In this problem you will determine a continued fraction expansion of the real number $\sqrt{2}$.

(a) Let $(a_n)$ be a sequence. Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$ then $(a_n)$ is convergent and $\lim_{n\to\infty} a_n = L$.

(b) Define the sequence $(a_n)$ where

$$a_1 = 1, \quad a_{n+1} = 1 + \frac{1}{1 + a_n}, \; n = 1, 2, 3, \ldots$$

i. Write down the first eight terms of $(a_n)$.

ii. Use part (a) to show that $(a_n)$ is convergent and $\lim_{n\to\infty} = \sqrt{2}$. Deduce the **continued fraction expansion**

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \cdots}}$$

C2. In this problem you will show the existence of *Euler's constant* $\gamma$. It is not know whether $\gamma$ is rational or irrational.

(a) Show that

$$\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n},$$

where $\log(x)$ is the natural logarithim function.

(b) Define

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n), \qquad n = 1, 2, 3, \ldots$$

Show that the sequence $(a_n)$ is decreasing and that $a_n \geq 0$, for each $n$. The limit $\gamma = \lim_{n\to\infty} a_n$ is known as *Euler's constant*, after Leonhard Euler (1707-1783).