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Problems for submission

A1. Consider the sequence (a_n), where $a_n = \frac{1}{n^3+10}$. In this problem you will show that (a_n) is convergent with limit $L = 0$ using the Squeeze Theorem.

(a) Carefully explain why the sequence (b_n), where $b_n = 0$, is convergent with limit $L = 0$.

(b) Carefully explain why the sequence (c_n), where $c_n = \frac{1}{n^3}$, is convergent with limit $L = 0$.

(c) Using the Squeeze Theorem, carefully explain why (a_n) is convergent with limit $L = 0$.

A2. We introduce the following definitions:

- a sequence (a_n) is **bounded above** if there exists M such that $a_n \le M$, for every $n \ge 1$. We call M an **upper bound** of (a_n).
- a sequence (a_n) is **bounded below** if there exists m such that $a_n \ge m$, for every $n \ge 1$. We call m a **lower bound** of (a_n).
- a sequence (a_n) is **alternating** if $a_n a_{n+1} < 0$, for every $n \ge 1$; that is, any two consecutive terms must have opposite sign.

For each of the following sequences (a_n)[1], determine which of the properties hold. You do not have to provide justification.

(i) bounded above, bounded below. If bounded above (resp. below) provide an explicit upper (resp. lower) bound.

(ii) increasing, decreasing or alternating.

(iii) convergent, divergent.

a) $a_n = \frac{2n^2}{n^2+1}$ b) $a_n = \frac{2n}{n^2+1}$ c) $a_n = \sin(1/n)$ d) $a_n = 4 - \frac{(-1)^n}{n}$ e) $a_n = \frac{n^2-(-1)^n}{n}$

f) $a_n = \frac{2^n}{(-\pi)^n}$ g) $a_n = \frac{5-2n}{n+5}$ h) $a_n = \frac{\sin(n)}{n}$ i) $a_n = \sqrt{n+1} - \sqrt{n}$ j) $a_n = \frac{(n!)^2}{(2n)!}$.

A3. Consider the sequence (a_n), where $a_n = \frac{1}{n^2+1}$. In this problem you will show *directly* that (a_n) is convergent with limit $L = 0$.

(a) Explain why $|a_n| = a_n$, for every natural number n.

(b) Determine a natural number N so that, if $n \ge N$ then $|a_n| < 20$.

(c) Determine a natural number N so that, if $n \ge N$ then $|a_n| < \frac{1}{10}$.

(d) Determine a natural number N so that, if $n \ge N$ then $|a_n| < 2^{-16}$.

(e) Let $\varepsilon > 0$. Determine a natural number N so that, if $n \ge N$ then $|a_n| < \varepsilon$. (*Hint: the natural number N will depend on ε.*)

A4. Consider the series $\sum_{n=1}^{\infty}(-1)^n$.

(a) Write down the first five partial sums s_1, s_2, s_3, s_4, s_5. What is a general expression for s_n?

(b) Is the series $\sum_{n=1}^{\infty}(-1)^n$ convergent or divergent? Explain your answer carefully.

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[1] Given a natural number n, define $n!$ (*n factorial*) to be the product $n! \overset{def}{=} 1 \cdot 2 \cdot 3 \cdots (n-2) \cdots (n-1) \cdot n$.

A5. For each of the following series determine whether the series converges or diverges; make sure you justify your conclusion. If the series converges determine its limit.

a) $\sum_{n=1}^{\infty} \frac{1}{5^n}$ b) $\sum_{n=1}^{\infty} 3\left(\frac{-1}{4}\right)^{n-1}$ c) $\sum_{n=0}^{\infty} \frac{5}{10^{3n}}$ d) $\sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}}$ e) $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}}$

f) $\sum_{n=1}^{\infty} \frac{3+2^n}{2^{n+2}}$ g) $\sum_{n=1}^{\infty} \frac{3+2^n}{3^{n+2}}$ h) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ i) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ j) $\sum_{n=1}^{\infty} \frac{n}{2n+1}$.

Additional recommended problems (not for submission)

B1. Let $f(x)$ be a differentiable function, defined for all $1 \leq x < \infty$ (and possibly on a larger domain). Consider the sequence $(a_n)_{n\geq 1}$, where $a_n = f(n)$.

(a) Suppose that $f'(x) \geq 0$, for all $1 \leq x < \infty$. Is the sequence (a_n) increasing, decreasing or neither?

(b) Suppose that $f'(x) \leq 0$, for all $1 \leq x < \infty$. Is the sequence (a_n) increasing, decreasing or neither?

(c) Consider the sequence (a_n), where $a_n = \cos\left(\frac{\pi}{n}\right)$. In this problem you will use your results above to show that (a_n) is a convergent sequence.

i. Let $f(x) = \cos\left(\frac{\pi}{x}\right)$. Show that $f'(x) = \frac{\pi \sin(\pi/x)}{x^2}$. (*Hint: chain rule!*)

ii. Explain why $\sin(\pi/x) \geq 0$ whenever $x \geq 1$.

iii. Explain carefully why (a_n) is an increasing sequence.

iv. Show that (a_n) is a bounded sequence and deduce that (a_n) is a convergent sequence.

v. What do you think the limit of the sequence (a_n) is? Justify your answer.

B2. Let (a_n) be a sequence satisfying $a_n > 0$, for every $n = 1, 2, 3, \ldots$. Complete the following statements:

(a) (a_n) is increasing is equivalent to $\frac{a_{n+1}}{a_n} \geq$ _____, for every $n = 1, 2, 3, \ldots$.

(b) (a_n) is decreasing is equivalent to $\frac{a_{n+1}}{a_n} \leq$ _____, for every $n = 1, 2, 3, \ldots$.

B3. Let (a_n) be a sequence.

- We say that (a_n) **diverges to** $+\infty$ if, for every real number K there exists a natural number N such that
$$n \geq N \quad \Longrightarrow \quad a_n > K.$$
We write, by abuse of notation, $\lim_{n\to\infty} a_n = +\infty$.

- We say that (a_n) **diverges to** $-\infty$ if, for every real number K there exists a natural number N such that
$$n \geq N \quad \Longrightarrow \quad a_n < K.$$
We write, by abuse of notation, $\lim_{n\to\infty} a_n = -\infty$.

(a) Determine whether the given sequence (a_n) diverges to $+\infty$, $-\infty$, neither.

a) $a_n = n$ b) $a_n = (-1)^n n$ c) $a_n = 2n + (-1)^n$ d) $\frac{n^2-4}{n+5}$ e) $a_n = \frac{(2n)!}{2(n!)}$

(b) Let c be a real number. Give an example of sequences (a_n), (b_n) such that $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = -\infty$ and $\lim_{n\to\infty}(a_n + b_n) = c$.

(c) Suppose that (a_n) is a sequence that diverges to $+\infty$.

i. Let (b_n) be a sequence such that $b_n \geq a_n$, for each $n = 1, 2, 3, \ldots$. Does (b_n) diverge to $+\infty$, $-\infty$,neither $\pm\infty$, or is there not enough information to decide?

ii. Let (b_n) be a sequence such that $b_n \leq a_n$, for each $n = 1, 2, 3, \ldots$. Does (b_n) diverge to $+\infty$, $-\infty$,neither $\pm\infty$, or is there not enough information to decide?

iii. Let (b_n) be a sequence such that $b_{2n} \geq a_{2n}$, for each $n = 1, 2, 3, \ldots$. Does (b_n) diverge to $+\infty$, $-\infty$,neither $\pm\infty$, or is there not enough information to decide?

B4. Consider the sequence (a_n), where

$$a_n = \frac{\alpha_r n^r + \alpha_{r-1} n^{r-1} + \ldots + \alpha_1 n + \alpha_0}{\beta_s n^s + \beta_{s-1} n^{s-1} + \ldots \beta_1 n + \beta_0}.$$

Here $\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_s$ are constants, $\alpha_r, \beta_s \neq 0$.

(a) Let $r < s$. Use the Limit Laws for Sequences to show that (a_n) is convergent with limit $L = 0$.

(b) Let $r = s$. Use the Limit Laws for Sequences to show that (a_n) is convergent with limit $L = \frac{\alpha_r}{\beta_s}$.

B5. (*For students who have seen mathematical induction*) In this problem you will determine an approach to approximating the real number $\sqrt{2}$. Let $a_1 = 1$, and define $a_{n+1} = \sqrt{1 + 2a_n}$, for $n = 1, 2, 3, \ldots$.

(a) Write down the first five terms a_1, a_2, a_3, a_4, a_5.

(b) Show that the sequence (a_n) is increasing. (*Hint: use induction*)

(c) Show that the sequence (a_n) is bounded above by 3. (*Hint: use induction*)

(d) Deduce that (a_n) is convergent. Let $L = \lim_{n\to\infty} a_n$ denote the (yet to be determined) limit of (a_n).

(e) Consider the sequence (b_n), where $b_n = \sqrt{1 + 2a_n}$. Explain carefully why $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$. Deduce that $L = 1 + \sqrt{2}$.

(f) Use the previous problem to describe an approach to determine an approximation of the real number $\sqrt{2}$ to within 10 decimal places.

B6. Is the following sequence convergent? If yes, determine its limit; if no, explain why not.

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

B7. Let (a_n) be a sequence.

- We say that (a_n) is **eventually increasing** if there is some natural number E such that the sequence $(a_n)_{n\geq E}$ is increasing.
- We say that (a_n) is **eventually decreasing** if there is some natural number F such that the sequence $(a_n)_{n\geq F}$ is decreasing.

Suppose that (a_n) is eventually increasing/eventually decreasing and bounded. Show that (a_n) is convergent.

B8. In this problem you will show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(a) Consider the series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ and denote its partial sums $s_1, s_2, s_3, \ldots$.

i. Write down the partial sums s_2, s_3, s_4, s_5, s_6.

ii. Show that $\sum_{n=2}^{m} \frac{1}{n(n-1)} = 1 - \frac{1}{m}$, for $m = 2, 3, 4, \ldots$.

iii. Conclude that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges and determine its limit.

(b) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and denote its partial sums $t_1, t_2, t_3, \ldots$. Show that $t_n \leq s_n$, for $n = 2, 3, 4, \ldots$ and deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (*Hint: show that the sequence $(t_m)_{m\geq 1}$ is increasing*)

Challenging Problems

C1. In this problem you will determine a continued fraction expansion of the real number $\sqrt{2}$.

(a) Let (a_n) be a sequence. Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$ then (a_n) is convergent and $\lim_{n\to\infty} a_n = L$.

(b) Define the sequence (a_n) where

$$a_1 = 1, \quad a_{n+1} = 1 + \frac{1}{1 + a_n}, \; n = 1, 2, 3, \ldots$$

i. Write down the first eight terms of (a_n).

ii. Use part (a) to show that (a_n) is convergent and $\lim_{n\to\infty} = \sqrt{2}$. Deduce the **continued fraction expansion**

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \cdots}}$$

C2. In this problem you will show the existence of *Euler's constant* γ. It is not know whether γ is rational or irrational.

(a) Show that

$$\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n},$$

where $\log(x)$ is the natural logarithim function.

(b) Define

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n), \qquad n = 1, 2, 3, \ldots$$

Show that the sequence (a_n) is decreasing and that $a_n \geq 0$, for each n. The limit $\gamma = \lim_{n\to\infty} a_n$ is known as *Euler's constant*, after Leonhard Euler (1707-1783).