Middlebury College

Calculus II: Fall 2017

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Practice Examination I

Instructions:

- You will have 60 minutes to complete this Examination.
- You *must* attempt Problem 1.
- Please attempt at least three of Problems 2, 3, 4, 5.
- If you attempt all five problems then your final score will be the sum of your score for Problem 1 and the scores for the three remaining problems receiving the highest number points.
- **Calculators are not permitted.**
- Explain your answers *clearly* and *nearly* and in *complete English sentences*. State all Theorems you have used from class. To recieve full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.
- Write your name on this exam and any extra sheets you hand in.

Question 1:	10/10
Question 2:	20/20
Question 3:	20/20
Question 4:	20/20
Question 5:	20/20
Total:	70/70

Name: Carl Friedrich Gauss

1. (10 points) True/False: you do not need to justify your solution.

   (a) Let (a_n) be a sequence. If the sequence of even terms $(a_2, a_4, a_6, \ldots)$ is convergent with limit L then (a_n) is convergent with limit L.

   (b) Let $\sum a_n$ be a series. If the associated sequence of partial sums (s_m) is decreasing and bounded then the sequence (a_n) is convergent.

   (c) Let (a_n) be a bounded sequence. Suppose that there exists N such that the sequence $(a_n)_{n \geq N}$ is decreasing. Then, (a_n) is convergent.

   (d) Let $\sum a_n$ be a series such that $a_n > 0$. Let (s_m) be the associated sequence of partial sums. If there exists K such that $s_m < K$, for $m = 1, 2, 3, \ldots$, then $\sum a_n$ is convergent.

   (e) Let $\sum (-1)^n b_n$, where $b_n > 0$, be an alternating series. If $\sum b_n$ is convergent then $\sum (-1)^n b_n$ is convergent.

Solution:

   (a) False: consider the sequence

$$(a_n) = (1, 0, 2, 0, 3, 0, 4, 0, 5, 0, \ldots)$$

   Then, $a_{2n} = 0$, for every n. In particular, the sequence (a_{2n}) is convergent. However, the sequence (a_n) is unbounded and therefore divergent.

   (b) True: if (s_m) is decreasing and bounded then (s_m) is convergent, by Monotonic+Bounded Theorem. Hence, the series $\sum a_n$ is convergent. Therefore, (a_n) is convergent (and $\lim a_n = 0$).

   (c) True: consider the sequence

$$(a_N, a_{N+1}, a_{N+2}, a_{N+3}, \ldots)$$

   This sequence is decreasing and bounded, hence convergent. Since the limit of a sequence is only dependent on its behaviour as $n \to \infty$ we have that the sequence

$$(a_1, a_2, \ldots, a_N, a_{N+1}, \ldots)$$

   is also convergent.

   (d) True: As the series has positive terms the sequence of partial sums (s_m) is a strictly increasing sequence. Also, for every m, $0 < s_m < K$ so that the sequence of partial sums is bounded. Hence, the sequence (s_m) is convergent by the Monotonic+Bounded Theorem. This means the series $\sum a_n$ is convergent.

   (e) Corrected! True: if $\sum b_n$ is convergent then $\sum (-1)^n b_n$ is absolutely convergent, hence convergent.

2. Determine if the following sequences converge or diverge. If the sequence converges determine the limit. Give a careful explanation of your solution.

   (a) (10 points)
$$\left(\frac{\sin(\frac{1}{n})}{2^n}\right)_{n\geq 1}$$

   (b) (10 points)
$$\left(\frac{n}{2+(-1)^n}\right)_{n\geq 1}$$

Solution:

(a) Let $b_n = -\frac{1}{2^n}$, $c_n = \frac{1}{2^n}$. Then, for all n we have
$$b_n \leq \frac{\sin(1/n)}{2^n} \leq c_n$$

Also, the sequences (b_n) and (c_n) are convergent and
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0.$$

This follows because if $-1 < x < 1$ then (x^n) is convergent with limit 0.

Hence, by the Squeeze Theorem the sequence $\left(\frac{\sin(1/n)}{2^n}\right)$ is convergent with limit 0.

(b) The first few terms of the sequence are
$$1, \frac{2}{3}, 3, \frac{4}{3}, 5, \frac{6}{3}, 7, \frac{8}{3}, \ldots$$

In particular, for any n, we have
$$\frac{n}{2+(-1)^n} \geq \frac{n}{3}.$$

Since the sequence $\left(\frac{n}{3}\right)$ is unbounded the same is true of the sequence $\left(\frac{n}{2+(-1)^n}\right)$. Hence, the sequence $\left(\frac{n}{2+(-1)^n}\right)$ is divergent.

3. (20 points) Consider the sequence (a_n), where

$$a_n = \frac{2^n}{n!}, \qquad n = 1, 2, 3, \ldots$$

(a) Show that (a_n) is a decreasing sequence.

(b) Determine an upper and lower bound for the sequence (a_n).

(c) Explain carefully why the series (a_n) is convergent.

(d) Determine $\lim_{n\to\infty} a_n$.

Solution:

(a) For $n = 1, 2, 3, \ldots$, we have

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2 \cdot 2^n}{(n+1)n!} = \frac{2}{(n+1)} \cdot \frac{2^n}{n!} = \frac{2}{(n+1)} a_n \le a_n$$

where the last inequality holds because $\frac{2}{n+1} \le 1$ whenever $n = 1, 2, 3, \ldots$.

(b) Obviously $a_n > 0$, for every n, so that 0 is a lower bound. Since the sequence is decreasing we have $a_n \le a_1 = 2$, for each n. Hence, 2 is an upper bound.

(c) The sequence (a_n) is decreasing and bounded. Hence, by the Monotonic+Bounded Theorem it is convergent.

(d) It seems like the sequence is converging to 0. Lets show this using the Squeeze Theorem. We let $b_n = 0$ and need to detemrine a candidate for c_n so that

$$b_n \le a_n \le c_n, \qquad n = 1, 2, 3, \ldots$$

Let's analyse a_n: we have

$$a_n = \frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \le \frac{2}{1} \cdot 1 \cdot 1 \cdot \cdots \cdot \frac{2}{n} = \frac{4}{n}$$

since $\frac{2}{3} \le 1$, $\frac{2}{4} \le 1, \ldots$, and $\frac{2}{n-1} \le 1$.

Let $c_n = \frac{4}{n}$. Then, (c_n) is convergent and $\lim_{n\to\infty} c_n = 0$. Similarly, (b_n) is convergent with limit 0. So, by the Squeeze Theorem we have $\lim a_n = 0$.

4. (20 points) Determine if the following series is convergent or divergent. If convergent you do not need to determine its limit. Justify your answer carefully.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{7^n - 2^n}$$

Solution: We will show that the series is absolutely convergent, hence it must be convergent. Now,

$$\left| \frac{\sin(n)}{7^n - 2^n} \right| \leq \frac{1}{7^n - 2^n}$$

We will show that the series $\sum_{n=1}^{\infty} \frac{1}{7^n - 2^n}$ is convergent. Hence, by the Direct Comparison Test the series $\sum \left|\frac{\sin(n)}{7^n-2^n}\right|$ is convergent, so that the series $\sum \frac{\sin(n)}{7^n-2^n}$ is absolutely convergent.
We will show that the series $\sum \frac{1}{7^n-2^n}$ is convergent by comparing it with the series $\sum \frac{1}{7^n}$. Indeed, since

$$\frac{1}{7^n - 2^n} \cdot \frac{7^n}{1} = \frac{7^n}{7^n - 2^n} = \frac{1}{1 - (\frac{2}{7})^n} \to \frac{1}{1-0} = 1, \quad \text{as } n \to \infty,$$

and the geometric series $\sum \frac{1}{7^n}$ is convergent (Geometric Series Theorem with $r = \frac{1}{7}$), the series $\sum \frac{1}{7^n-2^n}$ is convergent. The result follows.

5. (20 points) Determine whether the following series is absolutely convergent, conditionally convergent or divergent. If convergent you do not need to determine its limit. Justify your answer carefully.

$$\sum_{n=1}^{\infty}(-1)^{n+1}n\pi^{-n^2} = \pi^{-1} - 2\pi^{-4} + 3\pi^{-9} - \dots$$

Solution: First we check whether the series is absolutely convergent. We see that

$$\left|(-1)^{n+1}n\pi^{-n^2}\right| = \frac{n}{\pi^{n^2}}$$

Let's try to apply the Root Test and see what happens: we have

$$\sqrt[n]{\left|(-1)^{n+1}n\pi^{-n^2}\right|} = \sqrt[n]{\frac{n}{\pi^{n^2}}} = \sqrt[n]{n}\cdot\frac{1}{\pi^n} \to 1\cdot 0 = 0 < 1 \quad \text{as } n \to \infty.$$

Here we have used the Root Rule $\sqrt[n]{n} \to 1$ as $n \to \infty$, and the fact that $\frac{1}{\pi^n} \to 0$ as $n \to \infty$. Hence, by the Root Test the series is absolutely convergent.