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Infinite Products

In this note we develop the notion of an *infinite product*. Infinite products are analogs of series (i.e 'infinite sums'). The basic notions and some examples are discussed.

Let (b_n) be a sequence of nonzero real numbers.

1. The m^{th} **partial product associated to** (b_n) is

$$p_m = b_1 b_2 \cdots b_m.$$

2. The **sequence of partial products associated to** (b_n) is the sequence (p_m), where p_m is the m^{th} partial product associated to (b_n).

3. If the sequence (p_m) of partial products associated to (b_n) is convergent and $L = \lim_{m\to\infty} p_m \neq 0$, then we say that

$$\prod_{n=1}^{\infty} b_n = L = \lim_{m\to\infty} p_m.$$

We call $\prod_{n=1}^{\infty} b_n$ an **infinite product**. In this case, we say the infinite product $\prod_{n=1}^{\infty} b_n$ **converges**; otherwise, the infinite product **diverges**. In particular, if $\lim p_m = 0$ then the infinite product diverges.

Example:

1. Let $b_n = \frac{n-1}{n} = 1 - \frac{1}{n}$, for $n = 2, 3, 4, \ldots$. Then,

$$b_2 = \frac{1}{2},\ b_3 = \frac{2}{3},\ b_4 = \frac{3}{4}, \ldots$$

The partial products associated to (b_n) are

$$p_2 = b_2 = \frac{1}{2},\ p_3 = b_2 b_3 = \frac{1}{3},\ p_4 = b_2 b_3 b_4 = \frac{1}{4}, \ldots$$

In general,

$$p_m = b_1 b_2 \cdots b_m = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{m-1}{m} \cdot \frac{m}{m+1} = \frac{1}{m+1}$$

Hence, $\lim_{m\to\infty} p_m = 0$ and the infinite product diverges.

2. Let $b_n = \frac{n^2-1}{n^2} = 1 - \frac{1}{n^2}$, for $n = 2, 3, 4, \ldots$. Then,

$$b_2 = \frac{3}{4},\ b_3 = \frac{8}{9},\ b_4 = \frac{15}{16}, \ldots$$

The partial products associated to (b_n) are

$$p_2 = b_2 = \frac{3}{4},\ p_3 = b_2b_3 = \frac{2}{3},\ p_4 = b_2b_3b_4 = \frac{5}{8}, \ldots$$

To determine the m^{th} partial product we observe that, since $n^2 - 1 = (n-1)(n+1)$, we can write

$$\begin{aligned}
p_m &= \frac{(2-1)(2+1)}{2^2} \cdot \frac{(3-1)(3+1)}{3^2} \cdot \frac{(4-1)(4+1)}{4^2} \cdot \cdots \cdot \frac{(n-1)(n+1)}{n^2} \\
&= \frac{1 \cdot \cancel{3}}{2^{\cancel{2}}} \cdot \frac{\cancel{2} \cdot \cancel{4}}{\cancel{3^2}} \cdot \frac{\cancel{3} \cdot \cancel{5}}{\cancel{4^2}} \cdot \cdots \cdot \frac{\cancel{(m-2)m}}{\cancel{(m-1)^2}} \cdot \frac{\cancel{(m-1)}(m+1)}{m^{\cancel{2}}} \\
&= \frac{1}{2} \cdot \frac{m+1}{m}
\end{aligned}$$

Hence, as $m \to \infty$, $p_m \to \frac{1}{2}$. Thus, the sequence of partial products converges to $\frac{1}{2}$ so that

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$