Some solution to Induction Problems

① MATH 122 FALL 2017 MIDDLEBURY COLLEGE

Template: each problem that we solve below will have solution taking the following general form. Let P(n) be a mathematical proposition depending the natural number n.

Ⓐ <u>Base case</u>: Verify $P(1)$

Ⓑ <u>Inductive Step</u>: Assume $P(k)$, for some k. Want to show $P(k+1)$.

So, assume $P(k)$. Then,

$$\circledast$$

'here's where we have to do some maths'

Hence, $P(k+1)$

Ⓒ <u>Concluding Statement</u>: So, by mathematical induction, $P(n)$ for all n.

---

1) $P(n)$: $1^3 + 2^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$

Ⓐ $P(1)$: $1^3 = \dfrac{1^2(2)^2}{4}$ ✓

Ⓑ Assume $P(k)$: $1^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$

Then,

$$1^3 + \dots + k^3 + (k+1)^3 = (1^3 + \dots + k^3) + (k+1)^3$$
$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right)$$
$$= \frac{(k+1)^2}{4}\underbrace{\left(k^2 + 4k + 4\right)}_{(k+2)^2}$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$

Hence, $P(k+1)$.

ⓒ By mathematical induction, $P(n)$, for every n.

2) $P(n)$: $1.2 + 3.4 + 5.6 + \dots + (2n-1).2n = \dfrac{n(n+1)(4n-1)}{3}$

Ⓐ $P(1)$: $1.2 \stackrel{?}{=} \dfrac{1(1+1)(4.1-1)}{3} = \dfrac{1.2.3}{3}$ ✓

Ⓑ ~~Assume~~ $P(k)$: $1.2 + 3.4 + \dots + (2k-1).2k = \dfrac{k(k+1)(4k-1)}{3}$

Then;

$$1.2 + 3.4 + \dots + (2k-1).2k + (2(k+1)-1).2(k+1)$$
$$= (1.2 + 3.4 + \dots + (2k-1).2k) + (2k+1)2(k+1)$$
$$= \frac{k(k+1)(4k-1)}{3} + 2(2k+1)(k+1)$$
$$= \frac{1}{3}(k+1)\left[k(4k-1) + 6.(2k+1)\right]$$

$$= \frac{1}{3}(k+1)\left[4k^2 - k + 12k + 6\right]$$

$$= \frac{1}{3}(k+1)\underbrace{\left[4k^2 + 11k + 6\right]}_{(k+2)(4k+3)}$$

$$= \frac{1}{3}(k+1)(k+2)(4(k+1)-1)$$

Hence, $P(k+1)$.

(C) By induction, $P(n)$ for all n.

3)

$$P(n): \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \dots + \frac{n}{2} = \frac{n(n+1)}{4}.$$

(A) $P(1)$ $\quad \frac{1}{2} = \frac{1.(1+1)}{2}$ ✓.

(B) Assume $P(k)$: $\frac{1}{2} + \frac{2}{2} + \dots + \frac{k}{2} = \frac{k(k+1)}{4}$

Then;

$$\frac{1}{2} + \frac{2}{2} + \dots + \frac{k}{2} + \frac{(k+1)}{2}.$$

$$= \left(\frac{1}{2} + \frac{2}{2} + \dots + \frac{k}{2}\right) + \frac{(k+1)}{2}$$

$$= \frac{k(k+1)}{4} + \frac{k+1}{2}$$

$$= \frac{(k+1)}{4}\left[k + 2\right] = \frac{(k+1)(k+2)}{4}.$$

Hence, $P(k+1)$.

Ⓒ By induction, $P(n)$ for all n. (4)

7)

$P(n)$: $1.2^2 + 2.3^2 + 3.4^2 + \cdots + n(n+1)^2 = \frac{1}{12} n(n+1)(n+2)\cdot(3n+5)$

Ⓐ $P(1)$: $1 \cdot 2^2 = \frac{1}{12} 1.(1+1)(1+2)(3+5)$

$= \frac{1}{12} 1.(2)(3)(8)$ ✓

Ⓑ ~~Assume~~ $P(k)$:

$1 \cdot 2^2 + \cdots + k(k+1)^2 = \frac{1}{12} k(k+1)(k+2)\cdot(3k+5)$

Then,

$1 \cdot 2^2 + 2 \cdot 3^2 + \cdots + k(k+1)^2 + (k+1)(k+2)^2$

$= \left(1.2^2 + 2.3^2 + \cdots + k(k+1)^2\right) + (k+1)(k+2)^2$

$= \frac{1}{12} k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$

$= \frac{1}{12}(k+1)(k+2)\left[k(3k+5) + 12(k+2)\right]$

$= \frac{1}{12}(k+1)(k+2)\underbrace{\left[3k^2 + 17k + 24\right]}_{(k+3)(3k+8)}$

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$.

Hence, $P(k+1)$

Ⓒ By induction, $P(n)$ for every n.

8) Let $P(n)$: $3^n - 1$ divisible by 2.

(A) $P(1)$: $3^1 - 1 = 2$ ✓

(B) Assume $P(k)$: $3^k - 1$ divisible by 2.

$\Rightarrow 3^k - 1 = 2r$, for some integer r.

Then,

$$3^{k+1} - 1 = 3^k \cdot 3 - 1$$
$$= (2r+1)\,3 - 1$$
$$= 6r + 3 - 1$$
$$= 2(3r+1)$$

Hence, $3^{k+1} - 1$ divisible by 2.

ie $P(k+1)$ holds

(C) By induction, $P(n)$ for all n.

10) $P(n)$: $7^n - 1$ divisible by 6.

(A) $P(1)$: $7^1 - 1 = 6$. ✓

(B) Assume $P(k)$: $7^k - 1$ divisible by 6.

$\Rightarrow 7^k - 1 = 6r$, for some r.

Now, $7^{k+1} - 1 = 7^k \cdot 7 - 1$

$= (6r+1) \cdot 7 - 1 = 42r + 6$

$= 6 \cdot (7r+1)$

Hence, $P(k+1)$.

Ⓒ By induction, $P(n)$ for all n.

14) $P(n)$: $n^2 - 3n + 4$ is even.

Ⓐ $P(1)$: $1^2 - 3.1 + 4 = 2$ ✓

Ⓑ Assume $P(k)$: $k^2 - 3k + 4$ is even

$\Rightarrow k^2 - 3k + 4 = 2r$, for some r.

Then;

$(k+1)^2 - 3(k+1) + 4$

$= (k^2 - 3k + 4) + 2k + 1 - 3$

$= 2r + 2k - 2 = 2(r + k - 1)$

Hence, $P(k+1)$.

Ⓒ By induction, $P(n)$ for all n.