Middlebury College

Calculus II: Fall 2017

THURSDAY NOVEMBER 16

EXAMINATION II

READ THE FOLLOWING INSTRUCTIONS CAREFULLY

DO NOT OPEN THIS PACKET UNTIL INSTRUCTED

Instructions:

- Write your name on this exam and any extra sheets you hand in.
- Sign the Honor Code Pledge below.
- You will have 60 minutes to complete this Examination.
- You **must** attempt Problem 1.
- You **must** attempt **at least three** of Problems 2, 3, 4, 5.
- If you attempt all five problems then your final score will be the sum of your score for Problem 1 and the highest possible score obtained from three of the four remaining problems.
- There are 3 blank pages attached for scratchwork.
- **Calculators are not permitted.**
- **Explain your answers *clearly* and *neatly* and in *complete English sentences*.**
- State all Theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.

QUESTION 1:	/10
QUESTION 2:	/20
QUESTION 3:	/20
QUESTION 4:	/20
QUESTION 5:	/20
TOTAL:	/70

NAME: EMMY NOETHER

"I have neither given nor received unauthorized aid on this assignment"

1. (10 points) True/False. **You do not need to justify your solution.**

(a) For any $x > 0$,

$$\int_1^{x^3} \frac{dt}{t} = \int_1^{3x} \frac{dt}{t}$$

(b) Using the inverse trigonometric substitution $x = 3\tan(t)$, you determine

$$\int f(x)dx = 2\cos(t) + \tan(t)\sec(t) + \sin(2t) + C$$

Then,

$$f(x) = \frac{d}{dx}\left(\frac{6}{\sqrt{9+x^2}} + \frac{x\sqrt{9+x^2}}{9} + \frac{6x}{9+x^2}\right)$$

(c) Let $A = \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and let $f(x) = \cos\left(x - \frac{\pi}{3}\right)$ be the function having domain A. Then, the function $f(x)$ admits an inverse function.

(d)

$$\int \frac{1}{\sqrt{(1+x^4)^7}}dx = \frac{x}{\sqrt{(1+x^3)^5}} + C$$

(e) If $2x = \sin(t)$ then $\tan(4t) = \frac{4x}{\sqrt{1-4x^2}}$.

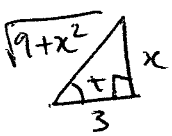
Solution:

(a) $\boxed{\text{FALSE}}$: $\log(x^3) = \int^{x^3} \frac{dt}{t} \neq \int^{3x} \frac{dt}{t} = \log(3x)$

in general

e.g. $\log(8) \neq \log(6)$.

(b)



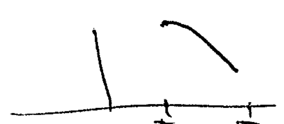
$x = 3\tan(t)$ $\quad$ $\sec(t) = \frac{\sqrt{9+x^2}}{3}$ $\quad$ $2\cos(t) + \tan(t)\sec(t) + \sin(2t)$

$\cos(t) = \frac{3}{\sqrt{9+x^2}}$ $\quad$ $\sin(t) = \frac{x}{\sqrt{9+x^2}}$ $\quad = \frac{6}{\sqrt{9+x^2}} + \frac{x\sqrt{9+x^2}}{9} + \frac{6x}{9+x^2}$

$\boxed{\text{TRUE}}$

(c)

Draw graph:

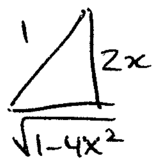


passes horizontal line Test

$\boxed{\text{TRUE}}$

(d) $\boxed{\text{FALSE}}$ $\quad \frac{d}{dx}\left(\frac{x}{(1+x^3)^{5/2}}\right) \neq \frac{1}{(1+x^4)^{7/2}}$

(e) $2x = \sin(t)$. $\quad \tan(2t) = \frac{\sin(2t)}{\cos(2t)}$



$= \frac{2\sin(t)\cos(t)}{\cos^2(t) - \sin^2(t)}$

$\cos(t) = \sqrt{1-4x^2}$

$\boxed{\text{FALSE}}$

$= \frac{4x\sqrt{1-4x^2}}{1-4x^2-4x^2} = \frac{4x\sqrt{1-4x^2}}{1-8x^2}$.

2. Determine the following antiderivative problems. Show all your working and provide complete justification.

(a) (10 points)

$$\int x \sec^2(x) dx$$

Solution:

Let $f = x \qquad g' = \sec^2(x)$

$\quad\; f' = 1 \qquad g = \tan(x)$

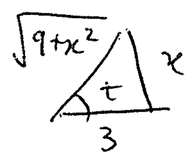
$$\int x \sec^2(x) dx = x \tan(x) - \int \tan(x) dx$$

$$= x \tan(x) - \log|\sec(x)| + C$$

(b) (10 points)

$$\int \frac{1}{\sqrt{(9+x^2)^3}} dx$$

Solution:



Let $x = 3\tan(t)$

$\quad \dfrac{dx}{dt} = 3\sec^2(t)$,

$$\frac{1}{\sqrt{(9+x^2)^3}} \cdot \frac{dx}{dt} = \frac{1}{\sqrt{(9(1+\tan^2(t)))^3}} \cdot 3\sec^2(t)$$

$$= \frac{1}{27} \cdot \frac{1}{\sec^3(t)} \cdot \sec^2(t)$$

$$= \frac{1}{27} \cdot \frac{1}{\sec(t)} = \frac{1}{27} \cdot \cos(t)$$

$$\Rightarrow \int \frac{1}{\sqrt{(9+x^2)^3}} dx = \frac{1}{27} \cdot \int \cos(t) dt$$

$$= \frac{1}{27} \sin(t) + C$$

$$= \frac{1}{27} \cdot \frac{x}{\sqrt{9+x^2}} + C .$$

3. Show all your working and provide complete justification.

(a) (10 points) Determine a partial fraction decomposition of the rational function

$$\frac{2x^2+4x+1}{x^3+2x^2+x}$$

Solution:

$$\frac{2x^2+4x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$2x^2+4x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

Input :

$x=0$ $\quad 1 = A$

$x=-1$ $\quad -1 = -C \Rightarrow C = 1$

$x=1$ $\quad 7 = 4 + 2B + 1 \Rightarrow B = 1$

$$\frac{2x^2+4x+1}{x(x+1)^2} = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

(b) Determine the antiderivative problem

$$\int \frac{2x^2+4x+1}{x^3+2x^2+x}dx$$

Solution:

$$\int \frac{2x^2+4x+1}{x^3+2x^2+x} dx$$

$$= \int \frac{1}{x} + \frac{1}{x+1} + \frac{1}{(x+1)^2} dx$$

$$= \log|x| + \log|x+1| - \frac{1}{(x+1)} + C$$

4. Let $n \geq 0$ and define the function $s_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$, where x is a real number.

(a) (10 points) Using induction, show that

$$\int s_n(x)dx = s_{n+1}(x) - 1 + C, \quad \text{for } n \geq 0.$$

Here C is a constant of integration.

Solution:

BASE CASE: $(n=0)$ $\quad \int s_0(x)dx = \int dx = x + C$

$= s_1(x) - 1 + C$ ✓

INDUCTIVE STEP:

Assume $\int s_k(x)dx = s_{k+1}(x) - 1 + C$.

Want to show: $\int s_{k+1}(x)dx = s_{k+2}(x) - 1 + C$.

Now, $\int s_{k+1}(x)dx = \int s_k(x) + \frac{x^{k+1}}{(k+1)!}\, dx$

$= \int s_k(x)dx + \int \frac{x^{k+1}}{(k+1)!} dx = s_{k+1}(x) - 1 + \frac{x^{k+2}}{(k+2)!} + C$

$= s_{k+2}(x) - 1 + C$.

Hence, by induction, $\int s_n(x)dx = s_{n+1}(x) - 1 + C$, for all $n \geq 0$.

(b) (10 points) Using integration by parts and part (a), show that

$$\int_0^1 x s_{n-1}(x)dx = 1 - \frac{1}{(n+1)!}, \quad n \geq 1$$

Solution:

Let $f = x \quad g' = s_{n-1}(x)$

$f' = 1 \quad g = s_n(x) - 1$

So,

$$\int_0^1 x s_{n-1}(x)\,dx = \Big[x(s_n(x) - 1)\Big]_0^1 - \int_0^1 (s_n(x) - 1)\,dx$$

$$= s_n(1) - 1 - \Big[s_{n+1}(x) - 1 - x\Big]_0^1$$

$$= s_n(1) - 1 - s_{n+1}(1) + 2$$

$$= 1 + s_n(1) - s_{n+1}(1)$$

Note: $s_n(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$s_{n+1}(1) = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$$\Rightarrow \int_0^1 x s_{n-1}(x)dx = 1 - \frac{1}{(n+1)!}$$

5. (a) (10 points) Let $f(x) = \sqrt{2x - x^2}$. Determine the surface area of the surface of revolution obtained by rotating $y = f(x)$ around the x-axis, $1 \le x \le 2$.

Solution:

$$A = 2\pi \int_1^2 f(x) \sqrt{1 + f'(x)^2}\, dx$$

$$f'(x) = \frac{1-x}{\sqrt{2x-x^2}}$$

$$1 + f'(x)^2 = 1 + \frac{(1-x)^2}{2x-x^2} = \frac{2x-x^2+(1-x)^2}{2x-x^2} = \frac{1}{2x-x^2}$$

$$= 2\pi \int_1^2 \sqrt{2x-x^2} \cdot \sqrt{\frac{1}{2x-x^2}}\, dx$$

$$= 2\pi \int_1^2 dx = 2\pi.$$

(b) Determine the arc length of the curve $y = \frac{x^2}{4} - \frac{1}{2}\log(x)$, where $1 \le x \le 2$.

Solution:

$$f(x) = \frac{x^2}{4} - \frac{1}{2}\log(x) \qquad L = \int_1^2 \sqrt{1 + f'(x)^2}\, dx$$

$$f'(x) = \frac{x}{2} - \frac{1}{2x}$$

$$f'(x)^2 + 1 = \left(\frac{x}{2} - \frac{1}{2x}\right)^2 + 1$$

$$= \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} + 1$$

$$= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$$

$$= \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$= \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^2 \frac{x}{2} + \frac{1}{2x}\, dx$$

$$= \left[\frac{x^2}{4} + \frac{1}{2}\log(x)\right]_1^2$$

$$= 1 + \frac{1}{2}\log(2) - \frac{1}{4}$$

$$= \frac{3}{4} + \frac{1}{2}\log(2).$$