Middlebury College

Calculus II: Fall 2017

THURSDAY OCTOBER 5

EXAMINATION I

READ THE FOLLOWING INSTRUCTIONS CAREFULLY

DO NOT OPEN THIS PACKET UNTIL INSTRUCTED

Instructions:

- Write your name on this exam and any extra sheets you hand in.
- Sign the Honor Code Pledge below.
- You will have 60 minutes to complete this Examination.
- You **must** attempt Problem 1.
- You **must** attempt at least three of Problems 2, 3, 4, 5.
- If you attempt all five problems then your final score will be the sum of your score for Problem 1 and the highest possible score obtained from three of the four remaining problems.
- There are 3 blank pages attached for scratchwork.
- **Calculators are not permitted.**
- **Explain your answers *clearly* and *neatly* and in *complete English sentences*.**
- State all Theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.

QUESTION 1:	10 /10
QUESTION 2:	20 /20
QUESTION 3:	20 /20
QUESTION 4:	20 /20
QUESTION 5:	20 /20
TOTAL:	70 /70

NAME: EVARISTE GALOIS

"I have neither given nor received unauthorized aid on this assignment"

Signature:

1. (10 points) True/False. **You do not need to justify your solution.**

   (a) Let (a_n) be a sequence. If $|a_n - 3| < \frac{1}{100}$, for all natural numbers n, then (a_n) is convergent with limit 3.

   (b) Let $\sum a_n$ be a series such that $a_n < 0$, for every n. If (a_n) is increasing then $\sum a_n$ is convergent.

   (c) Let (s_m) be sequence of partial sums associated to the series $\sum a_n$. Suppose that $-\frac{1}{n} \le \frac{s_m}{n} \le 3^{-n}$, for $n = 1, 2, 3, \ldots$. Then, $\sum a_n$ is convergent.

   (d) The series $\sum_{n=1}^{\infty} \frac{3}{\pi^n}$ is convergent.

   (e) Let (a_n) be a sequence. Suppose that $|a_n| \le \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Then, (a_n) is convergent.

Solution:

(a)

FALSE

(b)

FALSE

(c)

FALSE

(d)

TRUE

(e)

TRUE

2. Determine if the following sequences converge or diverge. If the sequence converges determine the limit. Give a careful explanation of your solution.

(a) (10 points)

$$\left(\frac{n^2}{n+1}\right)_{n\geq 1}$$

Solution:

Let $a_n = \dfrac{n^2}{n+1} = n \cdot \left(\dfrac{n}{n+1}\right) \geq n \cdot \dfrac{1}{2}$

$n \geq 1$

$\Leftrightarrow 2n = n+n \geq n+1$

$\Leftrightarrow \dfrac{n}{n+1} \geq \dfrac{1}{2}$

As $\frac{n}{2}$ unbounded the same is true of a_n. Hence, (a_n) divergent

(b) (10 points)

$$\left(\frac{n}{2n^2+(-1)^n}\right)_{n\geq 1}$$

Solution:

Let $a_n = \dfrac{n}{2n^2+(-1)^n}$.

As $\dfrac{2n^2+(-1)^n}{n} \geq \dfrac{2n^2-1}{n}$

$\Rightarrow \dfrac{n}{2n^2-1} \geq \dfrac{n}{2n^2+(-1)^n}$

Also, $2n^2+(-1)^n \geq 0$, for every n, so

$$\frac{n}{2n^2-1} \geq \frac{n}{2n^2+(-1)^n} \geq 0$$

Since $\dfrac{n}{2n^2-1} = \dfrac{\cancel{n^2}}{\cancel{n^2}} \cdot \dfrac{\left(\frac{1}{n}\right)}{2-1/n^2} \longrightarrow \dfrac{0}{2-0} = 0$

we have (a_n) convergent and $\lim a_n = 0$, by the Squeeze Theorem.

3. (20 points) Consider the sequence (a_n), where

$$a_n = \frac{(n+1)!(2n)!}{n!(2n+2)!}, \qquad n = 1, 2, 3, \ldots$$

(a) Show that (a_n) is a decreasing sequence.

(b) Determine an upper and lower bound for the sequence (a_n).

(c) Explain carefully why the series (a_n) is convergent.

(d) Determine $\lim_{n\to\infty} a_n$.

Solution:

Let $a_n = \dfrac{(n+1)!\,(2n)!}{n!\,(2n+2)!}$

$$= \frac{\cancel{1\cdot 2}\cdots\cancel{n}\cdot(n+1)\cdot\cancel{1\cdot 2}\cdots\cancel{2n}}{\cancel{1\cdot 2}\cdots\cancel{n}\;\cdot\;\cancel{1\cdot 2}\cdots\cancel{2n}(2n+1)(2n+2)}$$

$$= \frac{n+1}{(2n+1)(2n+2)} = \frac{n+1}{(2n+1)\,2\,(n+1)}$$

$$= \frac{1}{2}\cdot\frac{1}{2n+1}$$

(a) As $2(n+1)+1 = 2n+3 > 2n+1$

$$\Rightarrow \quad a_n = \frac{1}{2n+1} > \frac{1}{2(n+1)+1} = a_{n+1}$$

Hence, (a_n) decreasing.

(b) We have: upper bound (e.g.) 1 Any upper bound $\geq \frac{1}{6}$

lower bound (e.g.) 0 Any lower bound ≤ 0

(c) As (a_n) decreasing and bounded, by Monotonic + Bounded Theorem, (a_n) converges.

(d) We have $a_n = \dfrac{1}{2}\cdot\dfrac{1}{2n+1}$

$$= \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{1}{2+\frac{1}{n}} \longrightarrow \frac{1}{2}\cdot 0\cdot\frac{1}{2+0}$$

$$= 0$$

Hence, $\lim a_n = 0$.

4. (20 points) Determine if the following series is convergent or divergent. If convergent you do not need to determine its limit. Justify your answer carefully.

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right) \frac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1}$$

Solution:

Let $a_n = \sin\left(\frac{\pi}{n}\right) \dfrac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1}$

Then, $|a_n| = \left| \sin\left(\frac{\pi}{n}\right) \dfrac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1} \right|$

$\leq \dfrac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1}$

(*) If $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1}$ converges then, by DCT, $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges.

So, want to show $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1}$ converges

Apply LCT: let $b_n = \dfrac{1}{n^2}$. Then,

$$\frac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1} \cdot \frac{n^2}{1}$$

$$= \frac{\sqrt{n^2(4+\frac{2}{n}+1/n^2)}}{n^3(3+2/n+\frac{1}{n^2}+\frac{1}{n^3})} \cdot \frac{n^2}{1}$$

$$= \frac{\cancel{n^3}\sqrt{4+2/n+1/n^2}}{\cancel{n^3}(3+2/n+1/n^2+1/n^3)} \longrightarrow \frac{\sqrt{4+0+0}}{(3+0+0+0)} = \frac{2}{3} > 0$$

Hence, as $\sum \frac{1}{n^2}$ converges the same is true of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n^2+2n+1}}{3n^3+2n^2+n+1}$. So, by (*), $\sum a_n$ converges.

CONVERGENT

5. Determine whether the following series is absolutely convergent, conditionally convergent or divergent. If convergent you do not need to determine its limit. Justify your answer carefully.

(a) (10 points)

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n + 3^n}$$

Solution:

Let $a_n = \dfrac{3^n}{2^n+3^n} = \dfrac{\cancel{3^n}}{\cancel{3^n}} \cdot \dfrac{1}{(\frac{2}{3})^n+1}$

$\longrightarrow \dfrac{1}{0+1} = 1 \neq 0$

By Test for Divergence, $\sum a_n$ diverges.

DIVERGENT

(b) (10 points)

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n^2+1}$$

Solution:

As $\dfrac{2+(-1)^n}{n^2+1} = \begin{cases} \dfrac{1}{n^2+1} > 0 & n \text{ odd} \\ \dfrac{3}{n^2+1} > 0 & n \text{ even} \end{cases}$,

the series is absolutely convergent precisely when the series is convergent.

Note: $\dfrac{2+(-1)^n}{n^2+1} \leq \dfrac{3}{n^2+1}$, for all n.

$\leq \dfrac{3}{n^2}$, for all n.

Since $\sum_{n=1}^{\infty} \dfrac{3}{n^2}$ convergent (p-series, $p=2$), the same is true of $\sum_{n=1}^{\infty} \dfrac{2+(-1)^n}{n^2+1}$ by DCT.

Hence, series is absolutely convergent.