REVIEW PROBLEMS

Sequences and series

1) $0 < a_n = \dfrac{2^n}{n^n} = \left(\dfrac{2}{n}\right)^n \leq \dfrac{2}{n}$, for all n.

Since $\lim \dfrac{2}{n} = 0$, the Squeeze Theorem gives $\lim a_n = 0$

2) $$\frac{4^{n+1}}{10^{n-1}} = 4 \cdot \frac{4^n}{10^{n-1}} \frac{10}{10}$$

$$= 40 \cdot \left(\frac{2}{5}\right)^n$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4^{n+1}}{10^{n-1}} = 40 \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$$

$$= 40 \cdot \frac{\frac{2}{5}}{1-\frac{2}{5}}$$, Geometric series test

$$= 40 \cdot \frac{2}{3} = \frac{80}{3}$$

3) $\dfrac{-n}{n^3+1} \leq a_n \leq \dfrac{n}{n^3+1}$. As $\dfrac{n}{n^3+1}$

$$= \frac{n}{n^3} \cdot \frac{1}{1+\frac{1}{n^3}}$$

$$\longrightarrow 0 \cdot 1 = 0$$

Hence, using Squeeze Theorem, $\lim a_n = 0$

4) $a_n = 2^n \cos(\pi n)$

$= 2^n (-1)^n$

$= (-2)^n$

We saw in class, (x^n) convergent $\iff -1 < x \leq 1$

Hence, sequence is divergent.

5) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \sqrt{n}}{\sqrt{n^3+1}}$. Let $a_n = \dfrac{n^2+\sqrt{n}}{\sqrt{n^3+1}}$

$b_n = \sqrt{n}$

$$\frac{a_n}{b_n} = \frac{n^2+\sqrt{n}}{\sqrt{n^3+1}} \cdot \frac{1}{\sqrt{n}}$$

$$= \frac{n^2}{n^{3/2}} \cdot \frac{(1+n^{-3/2})}{\sqrt{1+1/n^3}} \cdot \frac{1}{\sqrt{n}}$$

$$= \frac{\left(1+\frac{1}{n^{3/2}}\right)}{\sqrt{1+1/n^3}} \longrightarrow \frac{1}{1} = 1$$

By LCT, since $\sum b_n$ divergent, $\sum a_n$ is also divergent

6) $\displaystyle\lim \frac{n}{4n-1} = \lim \frac{1}{4-1/n} = \frac{1}{4} \neq 0$

By Test for Divergence, series diverges

7) $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$ Let $a_n = \frac{n^5}{5^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^5}{5^{n+1}} \cdot \frac{5^n}{n^5}$$

$$= \frac{1}{5} \frac{(n+1)^5}{n^5} \longrightarrow \frac{1}{5} \text{ as } n \to \infty$$

By ratio test, series converges.

8) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+1}$

Note: $\lim_{n\to\infty} \frac{n}{n^3+1} = 0$ and $\left(\frac{n}{n^3+1}\right)$ decreasing

Let $f(x) = \frac{x}{x^3+1}$

$$f'(x) = \frac{1}{x^3+1} - \frac{3x^3}{(x^3+1)^2}$$

$$= \frac{x^3+1-3x^3}{(x^3+1)^2} = \frac{1-2x^3}{(1+x^3)^2} < 0 \text{ for } x \geq 1$$

Hence, $\frac{n}{n^3+1}$ strictly decreasing.

9) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ Let $a_n = \frac{(2n)!}{(n!)^2}$

$$= \frac{2n(2n-1)\ldots(n+1)\cancel{n}\cancel{(n-1)}\ldots\cancel{2\cdot 1}}{n(n-1)\ldots 1\ \cancel{n}\cancel{(n-1)}\ldots\cancel{2\cdot 1}}$$

$$> n+1$$

Hence, $\lim a_{n} \neq 0 \Rightarrow \sum a_n$ divergent.

1) $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 6}$ $n = 1, 2, 3, \ldots$

a) $P(n): a_{n+1} \geq a_n$

$P(1): a_2 = \sqrt{1+6} = \sqrt{7} > a_1 = 1$

Assume $P(k): a_{k+1} > a_k$

Then,

$$a_{k+2} = \sqrt{a_{k+1} + 6} > \sqrt{a_k + 6} = a_{k+1}$$

Hence, $P(k)$ holds. By math. induction, $P(n)$ for all n.

b) As (a_n) increasing, $a_n \geq a_1 = 1$, for all n.

Let $P(n): a_n < 3$.

Then, $P(1): a_1 = 1 < 3$ holds

Assume $P(k): a_k < 3$, for some k.

Then, $a_{k+1} = \sqrt{a_k + 6} < \sqrt{3+6}$

$$= \sqrt{9} = 3$$

Hence, $P(k+1)$ holds

By mathematical induction, $P(n)$ for all n.

c) Since (a_n) ~~is~~ increasing and bounded,

(a_n) is convergent.

Let $L = \lim_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim \sqrt{a_n + 6}$$

$$= \sqrt{\lim a_n + 6} = \sqrt{L+6}$$

$\Rightarrow \quad L^2 - L - 6 = 0$

Using quadratic formula,

$$L = \frac{1}{2} \pm \frac{\sqrt{1+24}}{2} = \begin{cases} 3 \\ -2 \end{cases}$$

Since $a_1 = 1 > 0$, we must have $L = 3$.

2) $a_1 = 7$, $a_{n+1} = \dfrac{a_n + 1}{4}$, $n = 1, 2, \ldots$

a) Let $P(n)$: $a_{n+1} \leq a_n$, $n = 1, 2, 3, \ldots$

Then, $P(1)$: $a_2 = \dfrac{7+1}{4} = 2 < a_1 = 7$ ✓

Inductive step: Assume $P(k)$: $a_{k+1} \leq a_k$.
Then,

$$a_{k+2} = \frac{a_{k+1} + 1}{4}$$
$$\leq \frac{a_k + 1}{4} = a_{k+1}$$

i.e $P(k+1)$ holds

Hence, by mathematical induction
$P(n)$, for all n.

b) As (a_n) decreasing $a_n \leq a_1 = 7$, for all n.

Now, let $P(n)$: $0 < a_n$.

Then, $P(1)$: $0 < 7$ ✓

Assume $P(k)$: $0 < a_k$, for some k

Then, $a_{k+1} = \dfrac{a_k + 1}{4} > \dfrac{1}{4} > 0$

$\Rightarrow$ $P(k+1)$ holds.

Hence, by mathematical induction
$P(n)$, for all n.

c) By monotonic + bounded theorem (a_n) convergent. Let $L = \lim_{n \to \infty} a_n$

Then, $L = \lim a_{n+1}$

$$= \lim \frac{a_n + 1}{4}$$

$$= \frac{L+1}{4}$$

$\Rightarrow$ $4L = L + 1$

$\Rightarrow$ $L = 1/3$

3) a Let $a_1 = 5$, $a_{n+1} = \sqrt{2a_n + 3}$, $n = 1, 2, 3, \ldots$

a) Let $P(n)$: $a_{n+1} \leq a_n$

Then, $P(1)$: $a_2 = \sqrt{2a_1 + 3} = \sqrt{13} < 5 = a_1$ ✓

Assume $P(k)$: $a_{k+1} = \sqrt{2a_k + 3}$, for some k.

Then,

$$a_{k+2} = \sqrt{2a_{k+1} + 3} \leq \sqrt{2a_k + 3} = a_{k+1}$$

using inductive step.

Hence, $P(k+1)$ holds.

Then, by mathematical induction, $P(n)$ for all n.

b) As (a_n) decreasing. $a_n \le a_1 = 5$,

for all n.

Now, let $P(n): 3 \le a_n$

Then, $P(1): 3 \le a_1 = 5$ ✓

Assume $P(k): 3 \le a_k$ for some k

Then,

$$a_{k+1} = \sqrt{2a_k + 3} \geq \sqrt{2.3+3}$$
$$= \sqrt{9} = 3.$$

Hence, $P(k+1)$ holds

By induction, $P(n)$ for all n.

c) By monotonic + bounded theorem sequence is convergent.

Let $L = \lim a_n$. Then.

$$\begin{aligned} L &= \lim a_n \\ &= \lim a_{n+1} \\ &= \lim \sqrt{2a_n + 3} \\ &= \sqrt{2 . \lim a_n + 3} \\ &= \sqrt{2L+3} \end{aligned}$$

$\Rightarrow L^2 = 2L + 3$

$\Rightarrow (L-3)(L+1) = 0$

Since $a_n \geq 3$ for all n, $L = 3$

(ie $L \neq -1$).

Techniques of integration

1) $$\frac{x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$\Rightarrow \quad x = A(x-1)^2 + B(x-1) + C$$

$$\Rightarrow \quad x = Ax^2 + (-2A+B)x + A - B + C$$

$$\Rightarrow \quad A = 0$$

$$-2A + B = 1 \quad \Rightarrow \quad B = 1$$

$$-B + C = 0 \quad \Rightarrow \quad C = 1$$

Hence, $$\frac{x}{(x-1)^3} = \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$\Rightarrow \quad \int \frac{x}{(x-1)^3}\,dx = \int \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}\,dx$$

$$= -\frac{1}{x-1} - \frac{1}{2}\frac{1}{(x-1)^2} + C$$

2) Let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$\Rightarrow \quad \int x^3 \cos(x^2)\,dx = \frac{1}{2}\int u\cos(u)\,du$$

$f = u \quad g' = \cos(u)$ $\qquad = \frac{1}{2}u\sin(u) - \int \sin(u)\,du$

$f' = 1 \quad g = \sin(u)$ $\qquad = \frac{1}{2}u\sin(u) + \cos(u) + C$

$$= \frac{1}{2}x^2\sin(x^2) + \cos(x^2) + C$$

3) Let $x = \sin(t)$

$$\frac{dx}{dt} = \cos(t)$$

$$\int \frac{\sqrt{1-x^2}}{x^4}\,dx = \int \frac{\cos^2(t)}{\sin^4(t)}\,dt$$

$f = \cos(t)$ $\quad g' = \frac{\cos(t)}{\sin^4(t)}$

$f' = -\sin(t)$ $\quad g = -\frac{1}{3}\frac{1}{\sin^3(t)}$

$$= \int \cos(t) \cdot \frac{\cos(t)}{\sin^4(t)}\,dt$$

$$= -\frac{1}{3}\frac{\cos(t)}{\sin^3(t)} - \frac{1}{3}\int \frac{1}{\sin^2(t)}\,dt$$

$$= -\frac{1}{3}\frac{\cos(t)}{\sin^3(t)} - \frac{1}{3}\int \frac{\cos^2(t)+\sin^2(t)}{\sin^2(t)}\,dt$$

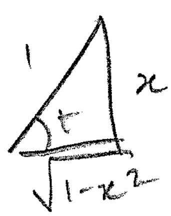
$f = \cos(t)$ $\quad g' = \frac{\cos(t)}{\sin^2(t)}$

$f' = -\sin(t)$ $\quad g = -\frac{1}{\sin(t)}$

$$= -\frac{1}{3}\frac{\cos(t)}{\sin^3(t)} - \frac{1}{3}\int \frac{\cos^2(t)}{\sin^2(t)} + 1\,dt$$

$$= -\frac{1}{3}\frac{\cos(t)}{\sin^3(t)} - \frac{1}{3}\left[-\frac{\cos(t)}{\sin(t)} - \cancel{\int dt} \cancel{+t}\right]$$

$$= -\frac{1}{3}\frac{\cos(t)}{\sin^3(t)} + \frac{1}{3}\frac{\cos(t)}{\sin(t)} + C.$$



$$= -\frac{1}{3}\frac{\sqrt{1-x^2}}{x^3} + \frac{1}{3}\frac{\sqrt{1-x^2}}{x} + C.$$

$$= -\frac{1}{3}\frac{\sqrt{1-x^2}}{x^3}\cdot\left[1 - x^2\right] + C$$

$$= -\frac{1}{3}\frac{(1-x^2)^{3/2}}{x^3} + C.$$

4) $$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \\ & x^3 - x \\ \hline & x \end{array}$$

$$\Rightarrow \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

$$\Rightarrow \int \frac{x^3}{x^2-1}\,dx = \int x + \frac{x}{x^2-1}\,dx$$

$$u = x^2-1 \quad = \frac{x^2}{2} + \frac{1}{2}\log(x^2-1) + C.$$

5) $$\int \frac{\sin^3(x)}{\cos^7(x)}\,dx$$

$$= \int \frac{(1-\cos^2(x))}{\cos^7(x)} \cdot \sin(x)\,dx \qquad u = \cos(x), \quad \frac{du}{dx} = -\sin(x)$$

$$= \int u^{-5} - u^{-7}\,du$$

$$= -\frac{1}{4}u^{-4} + \frac{1}{6}u^{-6} + C$$

$$= -\frac{1}{4} \cdot \frac{1}{\cos^4(x)} + \frac{1}{6} \cdot \frac{1}{\cos^6(x)} + C.$$

6) $\int \cos(3\log(x))\,dx$

Let $f = \cos(3\log(x))$ $\qquad g' = 1$

$f' = -\sin(3\log(x)) \cdot \frac{3}{x}$ $\qquad g = x$

$= x\cos(3\log(x)) + 3\int \sin(3\log(x))\,dx =$

$f = \sin(3\log(x))$ $\qquad g' = 1$

$f' = \cos(3\log(x)) \cdot \frac{3}{x}$ $\qquad g = x$

$$= x\cos(3\log(x)) + 3 \cdot \left[x\sin(3\log(x)) - 3 \cdot \int \cos(3\log(x))\,dx \right]$$

$$= x\cos(3\log(x)) + 3x\sin(3\log(x)) - 9\int \cos(3\log(x))\,dx$$

$$\Rightarrow 10\int \cos(3\log(x))\,dx = x\left(\cos(3\log(x)) + 3\sin(3\log(x))\right) + C$$

$$\Rightarrow \int \cos(3\log(x))\,dx = \frac{x}{10}\left(\cos(3\log(x)) + 3\sin(3\log(x))\right) + C.$$

7) $\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$

$\Rightarrow \quad 2x+3 = A(x+1) + B$

$\Rightarrow \quad A = 2 \qquad B = 1$

$$\Rightarrow \int \frac{2x+3}{(x+1)^2}\,dx = \int \frac{2}{(x+1)} + \frac{1}{(x+1)^2}\,dx$$

$$= 2\log(x+1) - \frac{1}{(x+1)} + C$$

8) $$\int \sqrt{5+4x-x^2}\,dx$$

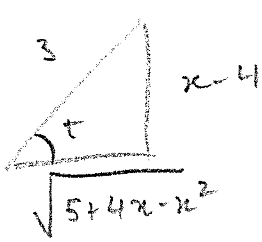
$$= \int \sqrt{9-(x-4)^2}\,dx$$

Let $x-4 = 3\sin(t)$

$$\frac{dx}{dt} = 3\cos(t)$$

$$= \int 9\cos^2(t)\,dt = 9\int \tfrac{1}{2}(1+\cos(2t))\,dt$$

$$= \frac{9}{2}\left[t + \tfrac{1}{2}\sin(2t)\right] + C$$



$$= \frac{9}{2}\arcsin\left(\frac{x-4}{3}\right) + \frac{1}{2}\sqrt{5+4x-x^2}\,.(x-4)$$

$$+ C.$$

$$\sin(2t) = 2\cos(t)\sin(t)$$

$$\cos(t) = \frac{\sqrt{5+4x-x^2}}{3}$$

$$\sin(t) = \frac{x-4}{3}.$$

9) $$\int \frac{x^2}{(5x^3-2)^{2/3}}\,dx$$

Let $u = 5x^3 - 2$

$$\frac{du}{dx} = 15x^2$$

$$\Rightarrow \int \frac{x^2}{(5x^3-2)^{2/3}}\,dx = \frac{1}{15}\int \frac{1}{u^{2/3}}\,du$$

$$= \frac{1}{15}\left[3u^{1/3}\right] + C$$

$$= \frac{1}{5}(5x^3-2)^{1/3} + C$$

10) $f = \arcsin\left(\frac{x}{2}\right)$ $\qquad g' = x$

$f' = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}$ $\qquad g = \frac{x^2}{2}$

$$\int x \arcsin(x/2)\,dx = \frac{x^2}{2}\arcsin\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$

$f = x$ $\qquad g' = \frac{x}{\sqrt{4-x^2}}$

$f' = 1$ $\qquad g = -\sqrt{4-x^2}$

$$= \frac{x^2}{2}\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} - \frac{1}{2}\int \sqrt{4-x^2}\,dx$$

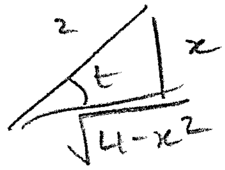
Let $x = 2\sin(t)$

$$\frac{dx}{dt} = 2\cos(t)$$

$$= \frac{1}{2}\int 4\cos^2(t)\,dt$$

$$= \int 1 + \cos(2t)\,dt$$

$$\underline{ie} \quad \frac{1}{2}\int\sqrt{4-x^2}\,dx \overset{x=2\sin(t)}{=} \int 1+\cos(2t)\ dt$$



$$= t + \frac{1}{2}\sin(2t)$$

$$= t + \cos(t)\sin(t)$$

$$= \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4} + C$$

$\Rightarrow$

$$\int x\arcsin\left(\frac{x}{2}\right)dx$$

$$= \frac{x^2}{2}\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} - \arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{4} + C$$

$$= \frac{x^2}{2}\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{4} - \arcsin\left(\frac{x}{2}\right) + C$$

11) $$\frac{x}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$\Rightarrow \quad x = (x^2+2)(Ax+B) + (x^2+1)(Cx+D)$$

$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + 2B+D$$

$$\Rightarrow \quad \left.\begin{aligned} A+C &= 0 \\ B+D &= 0 \\ 2A+C &= 1 \\ 2B+D &= 0 \end{aligned}\right\}$$

$A = -C \Rightarrow 2A = 1 - C = 1 + A$

$\Rightarrow A = 1, \ C = -1$

$B = -D \Rightarrow 2B = -D = B$

$\Rightarrow B = D = 0$

$$\int\frac{x}{(x^2+1)(x^2+2)}dx = \int\frac{x}{x^2+1} - \frac{x}{x^2+2}\ dx$$

$$= \int \frac{x}{x^2+1}\,dx - \int \frac{x}{x^2+2}\,dx$$

$$= \frac{1}{2}\log(1+x^2) - \frac{1}{2}\log(x^2+2) + C.$$

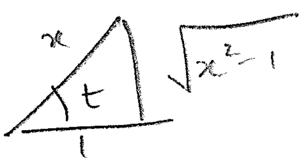
12) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$ Let $x = \sec(t)$

$$\frac{dx}{dt} = \sec(t)\tan(t)$$

$$= \int \frac{\cos^3(t)}{\tan(t)} \cdot \sec(t)\tan(t)\,dt$$

$$= \int \cos^2(t)\,dt = \frac{1}{2}\int 1 + \cos(2t)\,dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right] + C$$



$$= \frac{1}{2}\left[t + \cos(t)\sin(t)\right] + C$$

$$= \frac{1}{2}\operatorname{arcsec}(x) + \frac{1}{2}\,\frac{1}{x}\cdot\frac{\sqrt{x^2-1}}{x} + C$$

Improper integrals

1) $\displaystyle\int_0^\infty \frac{1}{x^2+x}\,dx$ Type I and II

infinite discontinuity at $x=0$.

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

ie $\displaystyle\int_0^\infty \frac{1}{x} - \frac{1}{x+1}\,dx = \int_0^1\left(\frac{1}{x} - \frac{1}{x+1}\right)dx + \int_1^\infty\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$

Now:

$$\int_0^1 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \lim_{t \to 0^+} \int_t^1 \frac{1}{x}\, dx - \int_0^1 \frac{1}{x+1}\, dx$$

$$= \lim_{t \to 0^+} \Big[\log(x)\Big]_t^1 - \Big[\log(x+1)\Big]_0^1$$

$$= -\lim_{t \to 0^+} \log(t) \qquad - (\log(2))$$

D.N.E

Hence, integral diverges.

2) $$\int_0^\infty \sin^2(s)\, ds$$

$$= \frac{1}{2} \int_0^\infty 1 - \cos(2s)\, ds$$

$$= \frac{1}{2} \lim_{t \to \infty} \int_0^t (1 - \cos 2s)\, ds$$

$$= \frac{1}{2} \lim_{t \to \infty} \left[s - \frac{1}{2}\sin(2s) \right]_0^t$$

$$= \frac{1}{2} \lim_{t \to \infty} \left(t - \frac{1}{2}\sin(2t) \right) \qquad \text{D.N.E}$$

Divergent

3) Type II :

$$\int_0^8 \frac{4}{(x-6)^3}\,dx = \int_0^6 \frac{4}{(x-6)^3}\,dx + \int_6^8 \frac{4}{(x-6)^3}\,dx$$

$$= \lim_{t \to 6^-} \int_0^t \frac{4}{(x-6)^3}\,dx$$

$$= \lim_{t \to 6^-} \left[-\frac{4}{2} \cdot \frac{1}{(x-6)^2} \right]_0^t$$

$$= \lim_{t \to 6^-} \left(\frac{1}{18} - \frac{2}{(t-6)^2} \right) \quad \text{D.N.E.}$$

Hence, divergent.

4) $$\int_0^\infty \frac{x \arctan(x)}{(1+x^2)^2}\,dx .$$

$$\frac{x \arctan(x)}{(1+x^2)^2} \leq \frac{\pi}{2} \cdot \frac{x}{(1+x^2)^2}, \text{ when } x \geq 0$$

Now,

$$\int \frac{x}{(1+x^2)^2}\,dx \overset{u=1+x^2}{=} \frac{1}{2}\int u^{-2}\,du$$

$$= -\frac{1}{2}u^{-1}.$$

$$= -\frac{1}{2}\,\frac{1}{1+x^2}$$

$$\Rightarrow \int_0^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{t \to \infty} \int_0^t \frac{x}{(1+x^2)^2}\,dx$$

$$= \lim_{t \to \infty} \left[\frac{1}{2} - \frac{1}{2}\,\frac{1}{1+t^2} \right]$$

$$= \frac{1}{2}$$

Hence, using comparison integral convergent

convergent

5) $$\int_0^\infty \frac{dx}{(2x+1)^3}\,dx .$$

$$= \lim_{t \to \infty} \int_0^t \frac{dx}{(2x+1)^3}$$

$$= \lim_{t \to \infty} \left[-\frac{1}{4}(2x+1)^{-2} \right]_0^t$$

$$= \lim_{t \to \infty} \left[\frac{1}{4} - \frac{1}{4}\,\frac{1}{(2t+1)^2} \right] = \frac{1}{4}$$

6) $$\int_0^\infty \frac{\arctan(x)}{2+\exp(x)}\,dx$$

__Note__: when $x \geq 0$

$$\arctan(x) \leq \frac{\pi}{2}$$

Hence: $$\frac{\arctan(x)}{2+\exp(x)} \leq \frac{\pi}{2}\cdot\frac{1}{2+\exp(x)}$$

$$\leq \frac{\pi}{2}\cdot\frac{1}{\exp(x)}$$

Also $$\int_0^\infty \exp(-x)\,dx$$

$$= \lim_{t\to\infty}\int_0^t \exp(-x)\,dx$$

$$= \lim_{t\to\infty}\Big[-\exp(-x)\Big]_0^t$$

$$= \lim_{t\to\infty}\Big[1-\exp(-t)\Big] = 1$$, converges.

Hence, $\frac{\pi}{2}\int_0^\infty \frac{1}{\exp(x)}\,dx$ convergent

$\Rightarrow \int_0^\infty \frac{\arctan(x)}{2+\exp(x)}\,dx$ Convergent using I.I.C.T.

7) $\displaystyle\int_{-2}^{4} \frac{1}{x^4}\,dx = \int_{-2}^{0} \frac{1}{x^4}\,dx + \underbrace{\int_{0}^{4} \frac{1}{x^4}\,dx}$

Type II

$$= \lim_{t \to 0^+} \int_{t}^{4} \frac{1}{x^4}\,dx$$

$$= \lim_{t \to 0^+} \left[-\frac{1}{3}\,\frac{1}{x^3} \right]_t^4$$

Divergent. D.N.E

8) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

Type II

$$= \lim_{t \to 1^-} \int_{0}^{t} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{t \to 1^-} \Big[\arcsin(x) \Big]_0^t$$

$$= \arcsin(1) = \pi/2$$

convergent

9) $\displaystyle\int_{0}^{5} \frac{t}{t-2}\,dt = \int_{0}^{2} \frac{t}{t-2}\,dt + \int_{2}^{5} \frac{t}{t-2}\,dt.$

$$= \lim_{s \to 2^+} \int_{s}^{5} 1 + \frac{2}{t-2}\,dt$$

$$= \lim_{s \to 2^+} \Big[t + 2\log(t-2) \Big]_s^5$$

Divergent. D.N.E.

10.) For $x \geq 0$

$$\frac{x}{x^3+1} \leq \frac{x}{x^3} = \frac{1}{x^2}$$

$$\int_0^\infty \frac{1}{x^2}\,dx = \int_0^1 \frac{dx}{x^2} + \int_1^\infty \frac{dx}{x^2}$$

D.N.E.

Hence, divergent.

<u>Differential Equations</u>

1) $y' - \frac{1}{x}y = \log(x)$ <u>Linear</u>

$P(x) = -1/x$ $\qquad \int P(x) = -\log(x)$

$Q(x) = \log(x)$ $\qquad \Rightarrow I(x) = \exp(-\log(x))$

$$= \exp(\log(x^{-1}))$$

$$= x^{-1}$$

$$\Rightarrow \quad I(x)\,y = \int Q(x) I(x)\,dx$$

$$= \int \frac{\log(x)}{x}\,dx$$

$u = \log(x)$

$$= \int u \,dx$$

$$= \frac{1}{2}(\log(x))^2 + C$$

$$\Rightarrow y = \frac{x}{2}(\log(x))^2 + xC$$

2. $f'(x) = f(x)\,(1-f(x))$

Let $y = f(x)$.

Separable

$$y' = y\,(1-y)$$

$$\int \frac{1}{y(1-y)}\,dy = \int dx$$

$$\Rightarrow \int \frac{1}{1-y} + \frac{1}{y}\,dy = x + c$$

$$\Rightarrow -\log(1-y) + \log(y) = x + c$$

$$\Rightarrow \log\left(\frac{y}{1-y}\right) = x + c$$

$$\Rightarrow \frac{y}{1-y} = A\exp(x),$$ A constant.

$$\Rightarrow \frac{y-1+1}{1-y} = A\exp(x)$$

$$\Rightarrow -1 + \frac{1}{1-y} = A\exp(x)$$

$$\Rightarrow 1-y = \frac{1}{1+A\exp(x)}$$

$$\Rightarrow y = 1 - \frac{1}{1+A\exp(x)}$$

$y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = 1 - \frac{1}{1+A}$

$$\Rightarrow A = 1$$

Hence, $y = f(x) = 1 - \dfrac{1}{1+\frac{1}{2}e^{x}}$

3) $\frac{dx}{dt} = 1 - t + x - tx$ Separable

$= (1-t)(1+x)$

$\Rightarrow \int \frac{dx}{1+x} = \int (1-t)\,dt$

$\Rightarrow \log(1+x) = t - \frac{t^2}{2} + C$

$\rightarrow 1+x = A \exp(t - t^2/2)$, A constant.

$\rightarrow x = A \exp(t - t^2/2) - 1$

4) $\frac{dr}{dt} + 2tr = r$

$\rightarrow \frac{dr}{dt} = r(1-2t)$ separable

$\Rightarrow \int \frac{dr}{r} = \int 1 - 2t \; dt$

$\rightarrow \log(r) = t - t^2 + C$

$\rightarrow r = A \exp(t - t^2)$

$5 = r(0) = A \Rightarrow r(t) = 5\exp(t - t^2)$

5) $y' = \frac{xy\sin(x)}{y+1} = \frac{y}{y+1} \cdot x\sin(x)$

Separable

$\Rightarrow \int \frac{y+1}{y}\,dy = \int x\sin(x)\,dx$

$= -x\cos(x) + \sin(x) + C$

$\Rightarrow \int \left(1 + \frac{1}{y}\right)dy = -x\cos(x) + \sin(x) + C$

$$\Rightarrow \quad y + \log(y) = -x\cos(x) + \sin(x) + c$$

$$\Rightarrow \quad \log(\exp(y)\cdot y) = -x\cos(x) + \sin(x) + c$$

$$\Rightarrow \quad \exp(y)\cdot y = A\exp(-x\cos(x) + \sin(x))$$

6) $y' + y\cos(x) = x\exp(-\sin(x))$

$P(x) = \cos(x)$ $\quad \int P(x) = \sin(x)$ <u>Linear</u>

$Q(x) = x\exp(-\sin(x))$ $\quad \Rightarrow I(x) = \exp(\sin(x))$

$$\Rightarrow I(x)y = \int I(x)\,Q(x)\,dx$$

$$= \int x\,dx = \frac{x^2}{2} + c$$

$$\Rightarrow \quad y = \frac{x^2}{2\exp(\sin(x))} + \frac{c}{\exp(\sin(x))}$$

7) <u>TYPO</u>

8) Separable

$$\int \frac{dy}{\exp(y)} = \int 3x^2\,dx = x^3 + c$$

$$\Rightarrow \quad -\exp(-y) = x^3 + c$$

$$\Rightarrow \quad y = \log\left(\frac{-1}{x^3+c}\right).$$

$$1 = y(0) = \log\left(\frac{-1}{c}\right)$$

$$c = -e^{-1}.$$

So, $$y = \log\left(\frac{1}{e^{-1} - x^3}\right).$$

9) $xy' = y$ Seperable

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log(y) = \log(x) + c$$

$$\rightarrow \boxed{y = Ax}$$

<u>Power series</u>

1) $$\sum_{n=0}^{\infty} \frac{(-x)^n}{n^2 5^n} \qquad c = 0$$

Let $b_n = \dfrac{(-x)^n}{n^2 5^n}$

Then,

$$\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim \left| \frac{x^{n+1}}{(n+1)^2 5^{n+1}} \frac{n^2 5^n}{x^n} \right|$$

$$= \lim \frac{|x|}{5} \cdot \frac{n^2}{(n+1)^2} = \frac{|x|}{5}$$

Hence, convergent when $-5 < x < 5$

Check endpoints:

$x = -5$: $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent. p-series

$x = 5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ convergent by AST.

2) Let $b_n = \dfrac{(x+2)^n}{(n+2)!}$ centre $\underline{\underline{c=-2}}$.

$$\lim \left| \frac{b_{n+1}}{b_n} \right| = \lim \left| \frac{(x+2)^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{(x+2)^n} \right|$$

$$= \lim \frac{|x+2|}{n+3} = 0 < 1, \text{ for every } x.$$

Hence, interval of convergence is $(-\infty, \infty)$

3)

3. Let $\quad b_n = \dfrac{(2x+1)^n}{n\,4^n} = \dfrac{2^n\,(x+\frac{1}{2})^n}{n \cdot 4^n} \qquad c = -\frac{1}{2}$

$$= \frac{(x+\frac{1}{2})^n}{2^n \cdot n}$$

Then, $\displaystyle \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{(x+\frac{1}{2})^{n+1}}{2^{n+1}\cdot(n+1)} \cdot \frac{2^n\, n}{(x+\frac{1}{2})^n}\right|$

$$= \lim_{n\to\infty} \frac{|x+\frac{1}{2}|}{2} \cdot \frac{n}{n+1}$$

$$= \frac{|x+\frac{1}{2}|}{2}$$

Converges when $\quad |x+\frac{1}{2}| < 2$

i.e. $\quad -2 < x+\frac{1}{2} < 2$

$$\Rightarrow \quad -\frac{5}{2} < x < \frac{3}{2}$$

At endpoints:

$(x = \frac{-5}{2})$: $\quad \displaystyle \sum_{n=1}^{\infty} \frac{(-4)^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

converges by A.S.T.

$(x = \frac{3}{2})$ $\quad \displaystyle \sum_{n=1}^{\infty} \frac{4^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, divergent by p-series.

$$\Rightarrow \boxed{-\frac{5}{2} \leq x < \frac{3}{2}}$$

4) $b_n = \dfrac{2^n\,(x-2)^n}{(n+2)!}$. $\quad$ Then, $\quad c = 2$

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}\,(x-2)^{n+1}}{(n+3)!} \quad \frac{(n+2)!}{2^n\,(x-2)^n}\right|$$

$$= \lim_{n\to\infty} \frac{2|x-2|}{(n+3)} = 0 < 1$$

Converges for all x i.e. I.O.C.

$$\boxed{(-\infty, \infty).}$$

5) $b_n = \dfrac{(x-1)^n}{(2n+1)\,3^n}$, $c = 1$

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(2n+3)\,3^{n+1}} \cdot \frac{3^n(2n+1)}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{3}|x-1| \cdot \frac{2n+1}{2n+3}$$

$$= \frac{1}{3}|x-1|$$

Hence, series converges when

$$\frac{1}{3}|x-1| < 1$$

i.e

$$-1 < \frac{1}{3}(x-1) < 1$$

$$\Rightarrow \quad -3 < x-1 < 3$$

$$\Rightarrow \quad -2 < x < 4.$$

Endpoints:

$(x=-2)$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)\,3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ Convergent by A.S.T.

$(x=4)$ $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(2n+1)\,3^n} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$

Since $n \geq 1 \iff 3n = n+n+n \geq 2n+1$

$\iff \frac{1}{2n+1} \geq \frac{1}{3n}$

and $\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent,

the D.C.T gives $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ divergent also.

Hence, $\sum_{n=0}^{\infty} \frac{1}{2n+1} = 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1}$ <u>divergent</u>.

$\Rightarrow \boxed{-2 \leq x < 4}$

6) $b_n = \frac{(2x)^n}{n} = \frac{2^n \cdot x^n}{n}$, $c = 0$.

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} \cdot x^{n+1}}{n+1} \cdot \frac{n}{2^n \cdot x^n} \right|$$

$$= \lim_{n\to\infty} 2|x| \cdot \frac{n}{n+1} = 2|x|$$

Hence, series converges when

$$2|x| < 1$$

$$\Rightarrow -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

<u>Endpoints</u>:

$(x = -\tfrac{1}{2})$. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, by AST

$(x = \tfrac{1}{2})$ $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent

$\Rightarrow \boxed{-\tfrac{1}{2} \leq x < \tfrac{1}{2}}$

7) $b_n = n^n x^n$, $c=0$

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = n \cdot |x| = +\infty$$

Hence, $\lim \sqrt[n]{|b_n|} > 1$, for any $x \neq 0$

And power series converges at $\underline{x=0}$.

8) $b_n = (-1)^n n 4^n x^n$, $c=0$.

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \sqrt[n]{n 4^n |x|^n}$$

$$= \lim_{n\to\infty} \sqrt[n]{n} \cdot 4 \cdot |x|$$

$$= 4|x|$$

Hence, converges when

$$4|x| < 1$$

$$\Rightarrow \quad -\frac{1}{4} < x < \frac{1}{4}$$

<u>Endpoints</u>

$x = -1/4$: $\sum (-1)^n n 4^n \left(-\frac{1}{4}\right)^n = \sum n$ DIVERGENT

$x = 1/4$: $\sum (-1)^n n 4^n \left(\frac{1}{4}\right)^n = \sum (-1)^n$ DIVERGENT

$$\rightarrow \quad \boxed{-\frac{1}{4} < x < \frac{1}{4}}$$

Taylor series

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$$

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + \cdots$$

1) $$\frac{x^2}{1+x^2} = x^2 \cdot \frac{1}{1+x^2}$$

$$= x^2 \cdot \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right)$$

$$= x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots$$

2) $$\frac{1}{4-x} = \frac{1}{4} \cdot \frac{1}{1-\frac{x}{4}}$$

$$= \frac{1}{4} \cdot \left(1 + \frac{x}{4} + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3 + \left(\frac{x}{4}\right)^4 + \cdots\right)$$

$$= \frac{1}{4} + \frac{x}{4^2} + \frac{x^2}{4^3} + \frac{x^3}{4^4} + \cdots$$

$$\log(4-x) = \int \frac{1}{4-x}\, dx$$

$$= \int \frac{1}{4} + \frac{x}{4^2} + \frac{x^2}{4^3} + \frac{x^3}{4^4} + \cdots \, dx$$

$$= \frac{1}{4} x + \frac{x^2}{2.4^2} + \frac{x^3}{3.4^3} + \frac{x^4}{4.4^4} + \cdots + C$$

At $x = 0$

$$\log(4) = C$$

$$\Rightarrow \log(4-x) = \log(4) + \frac{1}{4}x + \frac{x^2}{2.4^2} + \frac{x^3}{3.4^3} + \cdots$$

3) Note: $\dfrac{1}{(1-3x)^3} = \dfrac{1}{18}\dfrac{d^2}{dx^2}\left(\dfrac{1}{1-3x}\right)$

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + 81x^4 + \cdots$$

$$\frac{d}{dx}\left(\frac{1}{1-3x}\right) = 3 + 18x + 81x^2 + 4.81x^3 + \cdots$$

$$\frac{d^2}{dx^2}\left(\frac{1}{1-3x}\right) = 18 + 162x + 4.3.81x^2 + 5.4.243x^3 + \cdots$$

$$\Rightarrow \frac{1}{(1-3x)^3} = \frac{1}{18}\left(18 + 162x + 4.3.81x^2 + 5.4.243x^3 + \cdots\right)$$

$$= 1 + 9x + 54x^2 + 270x^3 + \cdots$$

4) $f(\pi) = 0$

$f'(\pi) = -1$

$f''(\pi) = 0$

$f'''(\pi) = 1$

$$f^{(n)}(\pi) = \begin{cases} 0 & n \text{ even} \\ (-1)^k & n = 2k-1 \end{cases}$$

Hence, Taylor series centered at $c = \pi$

$$-(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots$$

5) $f(\frac{\pi}{2}) = 1$

$f'(\frac{\pi}{2}) = 0$

$f''(\frac{\pi}{2}) = -1$

$f'''(\pi/2) = 0$

$$f^{(n)}(\pi/2) = \begin{cases} 0 & n \text{ odd} \\ (-1)^k & n = 2k+1 \end{cases}$$

$\Rightarrow$ Taylor series centred at $c = \pi/2$

$$1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots$$

6) $f(1) = 0$

$f'(1) = 5$

$f''(1) = 20$

$f'''(1) = 60$

$f''''(1) = 120$

$f'''''(1) = 120$

$f^{(n)}(1) = 0, \quad n \geq 6$

$$5(x-1) + 10(x-1)^2 + 10(x-1)^3 + 5(x-1)^4 + (x-1)^5.$$

7) $$\frac{1+x}{x-1} = -\frac{(1+x)}{1-x} = \frac{-1}{1-x} - \frac{x}{1-x}$$

$$= -1 - x - x^2 - x^3 - \cdots \quad - x(1 + x + x^2 + x^3 + \cdots)$$

$$= -1 - 2x - 2x^2 - 2x^3 - 2x^4 - \cdots$$

8) $f^{(n)}(-5) = \dfrac{n!}{6^{n+1}}$

$\Rightarrow$ Taylor series centred at $c=-5$ is

$$\sum_{n=0}^{\infty} \frac{n!}{6^{n+1}\, n!}(x+5)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{6^{n+1}}(x+5)^n$$

$$= \frac{1}{6} + \frac{1}{6^2}(x+5) + \frac{1}{6^3}(x+5)^2 + \cdots$$

OR $\dfrac{1}{1-x} = \dfrac{1}{6-(x+5)}$

$$= \frac{1}{6} \cdot \frac{1}{1-\frac{(x+5)}{6}}$$

$$= \frac{1}{6}\left(1 + \frac{x+5}{6} + \frac{(x+5)^2}{6^2} + \frac{(x+5)^3}{6^3} + \cdots\right)$$

Taylor's inequality.

1) $f(x) = \cos\left(\frac{x}{2}\right)$

$f'(x) = \frac{1}{2} - \sin\left(\frac{x}{2}\right)$

$f''(x) = \left(\frac{1}{2}\right)^2 - \cos\left(\frac{x}{2}\right)$

$\vdots$

$$|f^{(n)}(x)| = \begin{cases} \frac{1}{2^n}\left|\cos\left(\frac{x}{2}\right)\right| \\ \frac{1}{2^n}\left|\sin\left(\frac{x}{2}\right)\right| \end{cases} \leq \frac{1}{2^n}, \quad \text{for any } |x| \leq d, \text{ any } n.$$

Taylor $\Rightarrow$ $|R_n(x)| \leq \dfrac{1}{2^n} \cdot \dfrac{d^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$

Since d arbitrary, $\lim R_n(x) = 0$, for all x.

Hence, $f(x) =$ T-S centred at $c=0$, for all x.

2) $f^{(n)}(x) = \dfrac{2^n \cdot n!}{(3-2x)^{n+1}}$ (DIFFICULT... o_O)

$$= \frac{2^n \cdot n!}{2^{n+1} \cdot (3/2 - x)^{n+1}} = \frac{1}{2} \frac{n!}{(\frac{3}{2}-x)^{n+1}}$$

$$\Rightarrow \quad |f^{(n)}(x)| = \frac{1}{2} \frac{n!}{|\frac{3}{2}-x|^{n+1}}$$

Taylor series equals $f(x)$ on interval centred at $c=0$ ie of the form

$(-R, R)$

Must have $R \le 3/2$ since $f(x)$ undefined at $x = 3/2$.

Let $0 < d < 3/2$. Then, if $|x| \le d$

$$|f_n(x)| \le \frac{1}{2} \cdot \frac{n!}{(\frac{3}{2}-d)^{n+1}}$$

TAYLOR $\Rightarrow$ $|R_{n-1}(x)| \le \frac{1}{2} \frac{\cancel{n!}}{\cancel{n!}} \frac{d^n}{(\frac{3}{2}-d)^{n+1}}$

$$= \frac{1}{2(\frac{3}{2}-d)} \cdot \left(\frac{d}{\frac{3}{2}-d}\right)^n =$$

$$= \frac{1}{3-2d} \cdot \left(\frac{1}{\frac{3}{2d} - 1} \right)^n$$

As $d < 3/2 \Rightarrow 1 < \frac{3}{2d}$

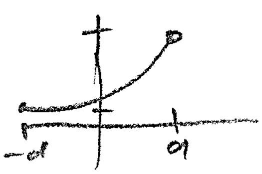
$$\Rightarrow \frac{1}{\frac{3}{2d} - 1} < 1$$

Hence, as $n \to \infty$, $\frac{1}{3-2d} \cdot \left(\frac{1}{\frac{3}{2d} - 1} \right)^n \to 0$.

$\Rightarrow \lim_{n \to \infty} R_{n-1}(x) = 0$, for $|x| \leq d < 3/2$

$\Rightarrow f(x) = \text{T.S.}$ for any x in $(-3/2, 3/2)$.

3) $f^{(n)}(x) = \exp(x+2)$.



Let $d > 0$. Then,

$$|f^{(n)}(x)| = \exp(x+2) \leq \exp(d+2), \quad |x| \leq d.$$

Since exp increasing.

TAYLOR $\Rightarrow$ $|R_{n-1}(x)| \leq \frac{\exp(d+2)}{n!} d^n, \quad |x| \leq d$

$\to 0$ as $n \to \infty$.

Hence, $\lim R_n(x) = 0$ for any x

$\Rightarrow f(x) = \text{T.S.}$ for any x.

Misc.

1) $\exp(-2) = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{2^n}{n!}$

$\Rightarrow \sum_{n=4}^{\infty} (-1)^n \frac{2^n}{n!} = \exp(-2) - 1 + 2 - \frac{2^2}{2!} + \frac{2^3}{3!}$

$= e^{-2} + \frac{1}{3}$

2) $\exp(3/5) = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{3^n}{5^n \cdot n!}$

3) $0 = \sin(\pi) = \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!}$

4) $\sum_{n=1} (-1)^n \frac{\log(2)^2}{n!} = \log(2)^2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

$= \log(2)^2 \cdot \left(e^{-1} - 1 \right)$

5) $-1 = \cos(\pi) = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

6) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x} \right) = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$\Rightarrow 4 = \frac{1}{(\frac{1}{2})^2} = \sum_{n=1}^{\infty} n \left(\frac{1}{2} \right)^{n-1}$