Middlebury College

Calculus II: Spring 2018

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Group Projects

Project Prompt:

The Middlebury College Museum of Art is having a special semester-long focus on calculus, titled *The Joy of Sequences & Series*. The Museum will display important items from the history of the development of calculus (e.g. Newton's wig, a painting created out of the first 100 million digits of π, a rectangular field having a fixed perimeter and largest area), and is expected to welcome many of the world's top mathematicians, scientists, artists, historians, musicians, and world leaders.

Due to your unparalleled command of derivatives, integrals, sequences and series, Middlebury College has chosen you, along with some of your fellow classmates, to produce a poster to be included in *The Joy of Sequences and Series*. Your poster will be on display during the exhibit and is intended to introduce the audience to an interesting piece of mathematics and provide some historical context.

Your poster must focus on one of three allowed themes and must include certain aspects to be considered for publication: these required topics are described in the Project Themes below. You should consider that your poster is to be read by the general population who may or may not have a college-level mathematical background. In particular, you will want to convey technical ideas and/or calculations in a down-to-earth manner, making sure you present the main ideas of a mathematical argument in an accessible manner. You may assume that readers are familiar with the mathematical notation seen in a standard college-level calculus course, however. You are also granted creative license to demonstrate an interesting application of your main topic, or expand on some historical or mathematical aspect of interest.

You will be provided with one 24" $\times$ 36" poster upon which you are free to paste text, drawings, diagrams, digital images etc. You are actively encouraged to include diagrams and visualisations to help make your report/poster interesting and approachable to the lay person and/or to emphasise a particular mathematical calculation or idea.

Grading:

Your final project score will consist of three components: *reflection* (15 points), *evaluation* (15 points) and *production* (30 points).

1. *Production* (30 points): This will depend solely on the report/poster produced by the group. A precise rubric outlining the grading of reports/posters will be posted at the Course Website.

2. *Reflection* (15 points): You will have to submit a 350-500 word summary outlining your contribution to the project and reflecting on what you learned and/or had difficulties with. You should also discuss your role in the group and reflect on how you interacted and contributed to the group process. A precise rubric outlining the grading this component of your score will be posted at the Course Website.

3. *Evaluation* ($\leq$ 15 points): You will have to evaluate your fellow group members and their contribution to the project. If all group members contribute equally to the group then each group member will receive 15 points. Otherwise, your group will have to come to a consensus and apportion your group's Evaluation points accordingly. An outline of how to do this will be posted at the Course Website.

Project themes

1. *Stirling's Formula*

Given a natural number n we define $n!$, pronounced n *factorial*, to be the product of the first n natural numbers

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

It can be shown that $n!$ grows faster than c^n, for any $c > 0$ - this means $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

The factorial function appears frequently in probability: for example, the probability of getting exactly n heads out of $2n$ tosses of a fair coin is

$$\frac{(2n)!}{2^{2n}(n!)^2}$$

In general, $n!$ can be computationally expensive to determine - 1000! has > 2500 digits...!

In his book *Methodus Differentialis* (1730), the British mathematician James Stirling derived the following approximation to $n!$

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n} \qquad (\textbf{Stirling's Formula})$$

where $e = \exp(1)$ is Euler's number. In fact, this approximation has a precise meaning:

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \sqrt{2\pi}$$

Therefore, Stirling's Formula gives a better approximation as n gets very large.

In this project you will derive Stirling's Formula. We will first analyse $\ln(n!)$ and then use what we've learned to determine Stirling's Formula.

Your project must contain the following elements:

(a) Compare Stirling's approximation with $n!$ for $n = 2, 5, 10, 100$. In particular, compare the number of digits.

(b) Define

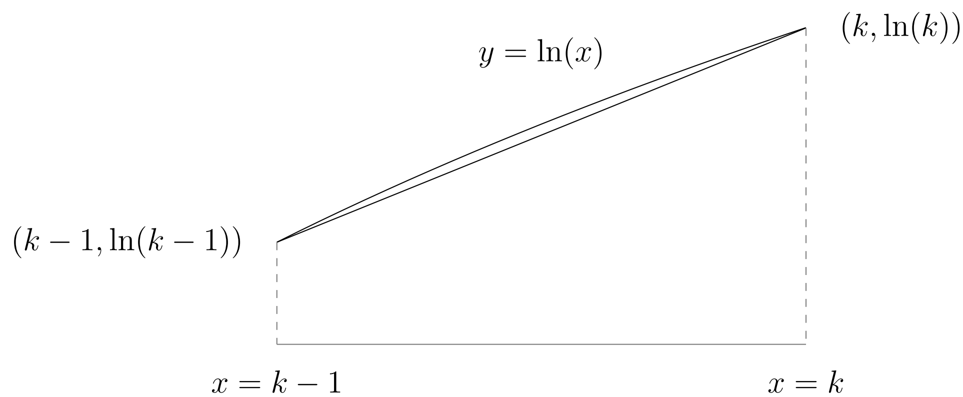
$$A_n = \ln(1) + \ln(2) + \ln(3) + \ldots + \ln(n-1) + \frac{1}{2}\ln(n) \qquad (1)$$

Observe that

$$\ln(n!) = \ln(1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n) = \ln(1) + \ln(2) + \ldots + \ln(n-1) + \ln(n) = A_n + \frac{1}{2}\ln(n)$$

We will define upper and lower bounds for the sequence (A_n) by approximating the area under the curve $y = \ln(x)$ by inscribed and circumscribed trapezoids.

(c) (**Upper bound**)

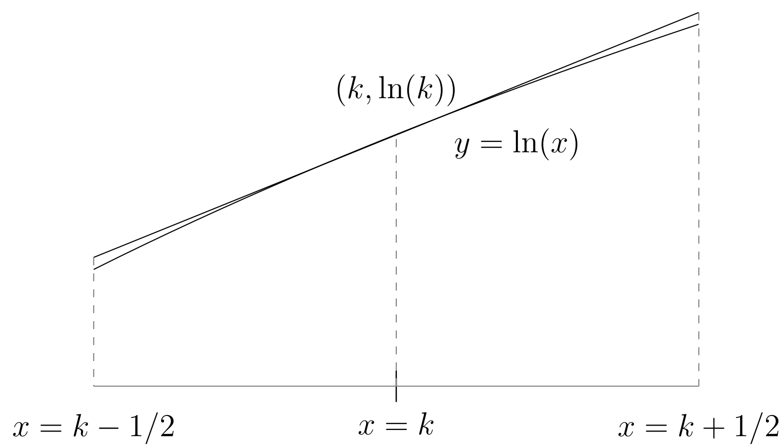


The straight line segment between $(k-1, \ln(k-1))$ and $(k, \ln(k))$ always lies beneath the graph $y = \ln(x)$ (why?). Show that the area of the trapezoid inscribed beneath the graph $y = \ln(x)$ between $x = k-1$ and $x = k$ has area $\frac{1}{2}(\ln(k-1) + \ln(k))$ and deduce that

$$\frac{1}{2}(\ln(1) + \ln(2)) + \frac{1}{2}(\ln(2) + \ln(3)) + \ldots + \frac{1}{2}(\ln(n-1) + \ln(n)) < \int_1^n \ln(x)dx \qquad (2)$$

Conclude that $A_n < \int_1^n \ln(x)dx$,

(d) (**Lower bound**)



Consider the tangent line to $y = \ln(x)$ at $x = k$. The equation of the tangent line is

$$y = \frac{x}{k} + \ln(k) - 1$$

Show that the area of the trapezoid beneath this tangent line between $x = k - 1/2$ and $x = k + 1/2$ is $\ln(k)$. Summing up the area of these trapezoids deduce that

$$\int_{3/2}^n \ln(x)dx < A_n$$

(e) Use inequalities (1) and (2) above to show that

$$n\ln(n) - n - \frac{3}{2}\ln\left(\frac{3}{2}\right) + \frac{3}{2} < A_n < n\ln(n) - n + 1$$

$$\implies \quad \frac{3}{2}\left(1 - \ln\frac{3}{2}\right) < \ln(n!) - \left(n + \frac{1}{2}\right)\ln(n) + n < 1$$

Evaluating $\frac{3}{2}\left(1 - \ln\frac{3}{2}\right)$ we obtain

$$0.8918 < \ln(n!) - \left(n + \frac{1}{2}\right)\ln(n) + n < 1$$

$$\implies \quad 2.395 < \frac{n!}{n^{n+1/2}e^{-n}} < 2.719, \quad \text{for any } n.$$

(f) We will now analyse the behaviour of $\frac{n!}{n^{n+1/2}e^{-n}}$. Write

$$\delta_n = \ln\left(\frac{n!}{n^{n+1/2}e^{-n}}\right) = \ln(n!) - \left(n + \frac{1}{2}\right)\ln(n) + n$$

and observe that $1 - \delta_n$ is the accumulated error from approximating the area under $y = \ln(x)$ by the inscribed trapezoids. Explain why this error is always positive (*Hint: the inscribed trapezoids always lie beneath the graph*), and deduce that $1 - \delta_n$ is increasing. Conclude that δ_n is a decreasing sequence and explain why $d = \lim_{n\to\infty} \delta_n$ exists.

(g) Define c so that $c = e^d$. Thus,

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = c$$

In particular, you've shown that, for n very large,

$$n! \sim cn^{n+1/2}e^{-n} \tag{3}$$

You will now show that $c = \sqrt{2\pi}$ - this will use *Wallis's Product Formula for* π. Wallis's Formula states that

$$\lim_{n\to\infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \cdots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) = \frac{\pi}{2}$$

(h) First, show the following identy:

$$2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2(n-1) \cdot 2n = 2^n n!$$

and use this to obtain

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots \cdot (2n-1) \cdot 2n}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot 2n} = \frac{(2n)!}{2^n n!}$$

Rewrite Wallis' Formula using these identities:

$$\lim_{n\to\infty} \frac{(2^n n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2}$$

(i) Use Stirling's approximation for $n!$ (equation (3)) to obtain

$$\lim_{n\to\infty} \frac{2^{4n}c^4 n^{4n+2}e^{-4n}}{c^2(2n)^{4n+1}e^{-4n}(2n+1)} = \frac{\pi}{2}$$

and deduce that $c = \sqrt{2\pi}$.

In addition to the above required topics, you must discuss at least one of the following topics:

(a') Show that the probability of getting exactly n heads from $2n$ tosses of a fair coin is approximately $\frac{1}{\sqrt{\pi n}}$, for n very large.

(b') Provide a brief biography of Stirling's life. (See Reference iv. for details).

(c') Any other topic of interest related to this project.

Resources: Unless otherwise indicated, the following resources are available in the Davis Family Library, or through the Library website. If you need help locating them then ask a College librarian (or me). Feel free to use further resources (there will be many!) to discover more about the life of Stirling and his contributions.

i. *Calculus in Context*, Ch. 12 (p. 770-774)
`http://www.math.smith.edu/ callahan/cic/book.pdf`

ii. C. Tweedie, *The Life of James Stirling, The Venetian.* The Math. Gazette, Vol. 10, No. 140, Jul. 1920 pp. 119-128.

iii. P. Maritz, *James Stirling: Mathematician and Mine Manager.* The Math. Intelligencer, Vol. 33, Iss. 3, Sep. 2011, pp.141-147.

2. *Infinite products and a formula for π:*

John Wallis was an English mathematician and contemporary of Isaac Newton who served as the chief cryptographer for the British parliament and Royal Court. Among his many contributions to mathematics, Wallis is credited with introducing the symbol ∞ for infinity. Wallis was a fierce champion of the English scientific community and engaged in several prolonged feuds with some of the preeminent scientists and philosophers of the time, including Hobbes, Fermat, Pascal and Descartes.

Wallis discovered a remarkable formula that can be used to approximate π, the **Wallis Product Formula**:

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots=\frac{\pi}{2}$$

The left hand side of the above expression is called an *infinite product*, and is a 'product version' of an infinite series. In general, given a sequence of nonzero real numbers (a_n), the m^{th} *partial product* is

$$p_m = a_1 a_2 \cdots a_m$$

If the associated *sequence of partial products* (p_m) converges to a nonzero limit then we define the **infinite product** to be the limit

$$\prod_{i=1}^{\infty} a_i \overset{def}{=} \lim_{m\to\infty} p_m$$

The theory of infinite products can be developed in a similar way to the theory of infinite series and is related to many familiar functions. For example, in the mid-1700s Euler guessed (correctly!) that the function $\sin(x)$ can be expressed in terms of infinite products:

$$\sin(x) = x\prod_{i=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right), \qquad \text{for any } x.$$

In this project you will investigate some of the basic theory of infinite products and outline a verification of the Wallis Product Formula for π.

Your project must contain the following elements.

(a) Let (b_n) be a sequence of nonzero real numbers. Discuss what it means for an infinite product $\prod_{n=1}^{\infty} b_n$ to *converge/diverge*. Determine the convergence/divergence of the *harmonic product* $\prod_{n=2}^{\infty}(1-\frac{1}{n})$ and $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$. See Reference i. for details.

(b) Define the *Wallis sequence* (w_n), where

$$w_n = \frac{4n^2}{4n^2-1} = 1+\frac{1}{4n^2-1}$$

Hence,

$$(w_n) = \left(\frac{4}{3}, \frac{16}{15}, \frac{36}{35}, \frac{64}{63}, \dots\right)$$

Explain why the Wallis Product Formula is equivalent to showing that the limit of the infinite product

$$\prod_{n=1}^{\infty} w_n = \frac{\pi}{2}$$

(c) Complete the following steps to determine the Wallis product formula (see Reference iii. for further details). For each natural number n, define $I_n = \int_0^{\pi/2} \sin^n(x)dx$.

i. Let n be a natural number. Explain why $0 \leq \sin^{n+1}(x) \leq \sin^n(x)$, for all $0 \leq x \leq \pi/2$.

ii. Use part i. to explain why the sequence $(I_1, I_2, I_3, \ldots)$ is decreasing. In particular,

$$I_{2n+2} \leq I_{2n+1} \leq I_{2n}$$

iii. Let n be a natural number. Using induction, show that

$$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1}(x)dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

and

$$I_{2n} = \int_0^{\pi/2} \sin^{2n}(x)dx = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \frac{\pi}{2}$$

and deduce that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(*Hint: use the reduction formulae from Problem Set 5*)

iv. Use the previous problems to show that

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1.$$

Explain why $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

v. Use the previous results to show that

$$\lim_{n\to\infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{\pi}{2}.$$

Deduce the Wallis Product Formula.

vi. Explain how you can rediscover the Wallis Product Formula from Euler's Product Formula for $\sin(x)$.

In addition to the above required topics, you must discuss at least one of the following topics:

(a') Use Wallis's Product Formula to approximate π. In your opinion, does the Product Formula provide an efficient approximation of π?

(b') Provide a brief account of the controversy between Wallis and the French mathematician Fermat. (See Reference iv. for details).

(c') Any other topic of interest related to this project.

Resources: Unless otherwise indicated, the following resources are available in the Davis Family Library. If you need help locating them then ask a College librarian (or me). Feel free to use further resources (there will be many!) to discover more about the life of Wallis and his interactions with other famous scientists of the 17th Century.

i. *Introductory notes on infinite products*. Available at Course Website.

ii. *The Wallis Product Formula for* π *and Its Proof*. Video, available at Course Website.

iii. D. J. Struik, Ed., *A Sourcebook in Mathematics 1200-1800*

iv. J. Stedall, *John Wallis and the French: his quarrels with Fermat, Pascal, Dulaurens and Descartes*

3. *The Bernoullis, Euler and the Basel Problem*

The Swiss-based Bernoulli family was one of the most famous families in the history of science. During the mid-late 17th Century and 18th Century, several Bernoullis (Jacob, Johann I, II, III, Nicolaus I, II) made significant contributions to mathematics and the physical sciences. In addition to their own contributions, the Bernoullis were well-known for popularising difficult *challenge problems* to the European scientific community at large. One of these problems (originally posed in 1644 by the Italian mathematician P. Mengoli) was to determine the limit of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

This problem became known as the *Basel problem* (after the location of the publishing house of Jacob Bernoulli). Several of the Bernoulli family made estimates for the limit

- Jacob: < 2,
- Johann/Daniel: $\sim \frac{8}{5}$.

In 1735, at the age of 28, Euler provided the first determination of this limit, and thereby solving the Basel problem, showing that

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$$

Miraculously, Euler's methods could be adapted to determine the limit of all p-series of the form $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$. However, an exact determination of $\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}}$ remains an unsolved problem to this day.

In this project you will investigate some of the contributions of the Bernoulli family to mathematics and Euler's solution to the Basel problem.

Your project must contain the following elements.

(a) Use a telescoping series argument to evaluate

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2.$$

Deduce, by a direct comparison, that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

(b) Consider the curves $y = x^n$, $0 \leq x \leq 1$,for $n = 0, 1, 2, 3, \ldots$. By considering the areas between successive curves, give a geometric demonstration of the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

(c) Explain, following p. 40-42 of Reference i., Jacob Bernoulli's determination of the series

$$\frac{a}{b} + \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \ldots = \frac{a}{b} + \sum_{n=1}^{\infty} \frac{a+nc}{bd^n}$$

Here a, b, c, d are integers, and we require $d > 1$. Use this to determine the limit of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

(d) To determine Euler's proof of the Basel problem you will need the following version of Newton's *Binomial Series Theorem* (which you can use without proof): for $|x| < 1$, k a fraction,

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{k(k-1)\cdots(k-(n-1))}{n!} x^n = (1-x)^k$$

Show that the Binomial Series Theorem implies that

$$(1-t^2)^{-1/2} = 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2^2 \cdot 2!}t^4 + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}t^6 + \ldots$$

(e) Suppose, as Euler did, that you are able to integrate functions defined by series *termwise* so that

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt = \int_0^x 1 dt + \int_0^x \frac{1}{2}t^2 dt + \int_0^x \frac{1 \cdot 3}{2^2 \cdot 2!}t^4 dt + \int_0^x \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}t^6 dt + \ldots$$

Explain why

$$\arcsin(x) = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots$$

(f) Use the Fundamental Theorem of Calculus to explain why

$$\frac{1}{2}(\arcsin(x))^2 = \int_0^x \frac{\arcsin(t)}{\sqrt{1-t^2}} dt$$

(g) Use integration by parts to determine the reduction formula

$$\int_0^1 \frac{t^{n+2}}{\sqrt{1-t^2}} dt = \frac{n+1}{n+2} \int_0^1 \frac{t^n}{\sqrt{1-t^2}} dt, \qquad n = 1, 2, 3, \ldots$$

(h) Use your computations above to outline, following p.55-57 of Reference i., Euler's 1741 proof of the Basel problem.

In addition to the above required topics, you must discuss at least one of the following topics.

(a') Euler's original 1734 proof of the Basel problem was considered somewhat mysterious, and was criticized for containing incomplete arguments that utilised an infinite product expansion for $\sin(x)$. Explain briefly Euler's product expansion for $\sin(x)$ and why Euler's use of this product expansion was consider incomplete. (See Reference i. p.46 and Reference ii.).

(b') Outline some of the contributions of the Bernoulli family to mathematics and the physical sciences: for example, the *Brachistrone problem*, the *Bernoulli principle* (fluid dynamics), the *Bernoulli distribution* (probability), or anything else!

(c') Any other topic of interest related to this project.

Resources: Unless otherwise indicated, the following resources are available in the Davis Family Library. If you need help locating them then ask a College librarian (or me). Feel free to use further resources (there will be many!) to discover more about the life of the Bernoullis, Euler and other famous scientists of the 17/18th Century.

i. W. Dunham, *Euler: The Master of Us All*
ii. E. Sandifer, *How Euler Did It* Available at Course Website.
iii. D. J. Struik, Ed., *A Sourcebook in Mathematics 1200-1800*
iv. E. T. Bell *Men of Mathematics*, Ch. 8 (*The Bernoullis*), Ch. 9 (*Euler*)