PROBLEMS ON SERIES: SOLUTIONS

1) $\sum_{n=1}^{\infty} \frac{1}{\pi^n+5}$: DCT, by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{\pi^n}$:

for $n=1,2,\ldots$, $\pi^n+5 > \pi^n \Rightarrow \frac{1}{\pi^n} > \frac{1}{\pi^n+5}$

Hence, apply DCT.

2) $\sum_{i=1}^{\infty} \frac{1+(-1)^i}{\sqrt{i}}$: Let $a_i = \frac{1+(-1)^i}{\sqrt{i}}$

$a_1 = 0$
$a_2 = \frac{2}{\sqrt{2}}$
$a_3 = 0$
$a_4 = \frac{2}{\sqrt{4}}$
$a_5 = 0$
$a_6 = \frac{2}{\sqrt{6}}$

The sequence of partial sums is:

$S_1 = 0$

$S_2 = \frac{2}{\sqrt{2}}$

$S_3 = \frac{2}{\sqrt{2}}$

$S_4 = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{4}}$

$S_5 = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{4}}$

$S_6 = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}$

$\Rightarrow$ For each $k=1,2,\ldots$

$S_{2k} = 2 \cdot \sum_{j=1}^{k} \frac{1}{\sqrt{2j}} = \frac{2}{\sqrt{2}} \sum_{j=1}^{k} \frac{1}{\sqrt{j}}$

$S_{2k+1} = S_{2k}$.

The sequence (S_m) is unbounded:

for even m

$S_m > \sum_{j=1}^{m/2} \frac{1}{\sqrt{j}}$

As $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}$ is divergent, its sequence of partial sums is <u>unbounded</u>. Hence, same is true for (S_m).

$\Rightarrow$ <u>DIVERGENT</u>

3) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$: Try ratio test: $a_n = \frac{(-3)^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!}$$

$$= 3 \cdot \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}\cdot(n+1)}$$

$$= \frac{3}{n+1} \xrightarrow{n\to\infty} 0 < 1$$

Hence, by Ratio Test $\sum a_n$ convergent.

4) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$: Try AST: let $b_n = \frac{1}{n^n}$

- $b_{n+1} = \frac{1}{(n+1)^n}$

by Binomial Theorem $\longrightarrow = \frac{1}{n^n + \binom{n}{1}n^{n-1} + \cdots + 1}$

$$< \frac{1}{n^n} = b_n$$

Hence, (b_n) decreasing.

- Since $0 < b_n = \frac{1}{n^n}$

$$= \frac{1}{n\cdot n\cdots n}$$

$$\leq \frac{1}{n}$$

By Squeeze Theorem, $\lim_{n\to\infty} b_n = 0$

Hence, AST implies series convergent

5) $\sum_{n=1}^{\infty} \frac{2+n}{n^3}$: LCT; Let $a_n = \frac{2+n}{n^3}$, $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{2+n}{n^3} \cdot \frac{n^2}{1} = \frac{2+n}{n} = \frac{2}{n} + 1 \xrightarrow{n\to\infty} 0+1 = 1 > 0$$

Hence, by LCT, since the series $\sum \frac{1}{n^2}$ is convergent (p-series, $p=2$) the same is true of $\sum_{n=1}^{\infty} \frac{2+n}{n^3}$. $\underline{\text{CONVERGENT}}$

6) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2-2}$. Let $a_n = \frac{n^2+1}{n^2-2} = \frac{\cancel{n^2}}{\cancel{n^2}} \frac{(1+\frac{1}{n^2})}{1-\frac{2}{n^2}}$

Then $\lim \frac{(1+\frac{1}{n^2})}{(1-\frac{2}{n^2})} = \frac{1+\lim \frac{1}{n^2}}{1-\lim \frac{2}{n^2}}$

$= \frac{1+0}{1-0} = 1 \neq 0$

Hence, by Test for Divergence, the series is $\underline{\text{DIVERGENT}}$.

7) $\sum_{m=1}^{\infty} \frac{\sin(4m)}{m^2}$: Let $a_m = \frac{\sin 4m}{m^2}$

Then, $|a_m| = \frac{|\sin 4m|}{m^2} \leq \frac{1}{m^2}$

Hence, by DCT with the convergent series $\sum \frac{1}{m^2}$, the series $\sum |a_m|$ is convergent.

Thus, $\sum a_m$ is absolutely convergent.

$\Rightarrow \sum a_m$ $\underline{\text{convergent}}$.

8) $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot 2n}{n!}$; let $a_n = \frac{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot 2n}{n!}$

Note: $a_n = \frac{(2 \cdot \cancel{1}) \cdot (2 \cdot \cancel{2}) \cdot (2 \cdot \cancel{3}) \cdots (2(\cancel{n-1}))(2\cancel{n})}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdots (\cancel{n-1}) \cdot \cancel{n}}$

$= 2^n \longrightarrow +\infty$ as $n \to \infty$

Hence, by Test for divergence, the series $\sum a_n$ is <u>divergent</u>

9) $\sum_{n=2}^{\infty} \frac{n}{2^n}$: Try ratio test: let $a_n = \frac{n}{2^n}$.

Then, $\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{n}{2^n}\right|} = \left(\frac{n}{2^n}\right)^{1/n}$

$$= \frac{\sqrt[n]{n}}{2} \xrightarrow{n\to\infty} \frac{1}{2} < 1$$

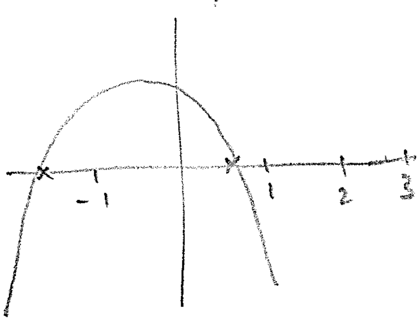
using Root Rule $\sqrt[n]{n} \xrightarrow{n\to\infty} 1$.

Hence, by ratio test, the series converges

10) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$: Try AST: let $b_n = \frac{n}{n^2+1}$

$f(x) = -x^2 - x + 1$

roots: $x = -\frac{1}{2} \pm \frac{\sqrt{1+4}}{2}$



• $b_{n+1} - b_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}$

$$= \frac{(n+1)(n^2+1) - n\left((n+1)^2+1\right)}{(n^2+1)\left[(n+1)^2+1\right]}$$

$$= \frac{\cancel{n^3} + n^2 + n + 1 - \cancel{n^3} - 2n^2 - 2n}{(n^2+1)\left[(n+1)^2+1\right]}$$

$$= \frac{1-n-n^2}{(n^2+1)\left[(n+1)^2+1\right]} \le 0 \text{, for } n = 1, 2, \ldots$$

Hence, (b_n) decreasing.

• Also, $\frac{n}{n^2+1} = \frac{n}{n^2} \cdot \frac{1}{1+\frac{1}{n^2}} = \frac{1}{n} \cdot \frac{1}{1+\frac{1}{n^2}}$

$$\lim \frac{n}{n^2+1} = \lim\left(\frac{1}{n} \cdot \frac{1}{1+\frac{1}{n^2}}\right)$$

$$= \left(\lim \frac{1}{n}\right) \cdot \left(\frac{1}{1+\lim\frac{1}{n^2}}\right) = 0 \cdot \left(\frac{1}{1+0}\right) = 0$$

Hence, by AST, $\sum (-1)^n b_n$ is convergent.

11) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2n+(-1)^n}$ Let $a_n = \dfrac{2}{2n+(-1)^n}$

for $n = 1, 2, 3, \ldots$

$$\frac{2n+(-1)^n}{2} \leq \frac{2n+1}{2} = n + \frac{1}{2} \leq n+1$$

Hence, $\dfrac{1}{n+1} \leq \dfrac{2}{2n+(-1)^n}$.

By DCT with divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1} = \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) - 1$

the series $\displaystyle\sum \frac{2}{2n+(-1)^n}$ is divergent.

12) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{3^n+5^n}$: Let $a_n = \dfrac{2^n+4^n}{3^n+5^n}$.

"LOOKS LIKE" (as n gets large)

$b_n = \dfrac{4^n}{5^n}$.

Then,

$$\frac{a_n}{b_n} = \frac{2^n+4^n}{3^n+5^n} \cdot \frac{5^n}{4^n}$$

$$= \frac{\cancel{4^n}\left(\left(\frac{2}{4}\right)^n+1\right)}{\cancel{5^n}\left(\left(\frac{3}{5}\right)^n+1\right)} \cdot \frac{\cancel{5^n}}{\cancel{4^n}}$$

$$= \frac{\left(\frac{1}{2}\right)^n+1}{\left(\frac{3}{5}\right)^n+1} \xrightarrow{n \to \infty} \frac{0+1}{0+1} = 1 > 0.$$

Hence, by LCT with convergent geometric series $\sum \left(\frac{4}{5}\right)^n$, the series $\sum a_n$ is convergent.

13) $\sum_{j=1}^{\infty} \frac{1}{2+\sin j}$: Let $a_j = \frac{1}{2+\sin j}$.

As $2 + \sin j \leq 2 + 1 = 3$

$\Rightarrow \quad \frac{1}{3} \leq \frac{1}{2+\sin j}$, for all j.

In particular, not possible that $\lim a_j = 0$

Hence; by Test for divergence, the series is divergent.

14) $\sum_{n=1}^{\infty} \frac{3^{n+2} n^3}{n!}$: try ratio test:

let $a_n = \frac{3^{n+2} n^3}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+3} \cdot (n+1)^3}{(n+1)!} \cdot \frac{n!}{3^{n+2} n^3}$$

$$= \frac{3^{n+3}}{3^{n+2}} \cdot \frac{(n+1)^3}{n^3} \cdot \frac{\cancel{1} \cdot \cancel{2} \cdots \cancel{n}}{\cancel{1} \cdot \cancel{2} \cdots \cancel{n}\,(n+1)}$$

$$= 3 \frac{\cancel{n^3}}{\cancel{n^3}} \left(1 + \frac{1}{n}\right)^3 \cdot \frac{1}{(n+1)}$$

$$\xrightarrow{n \to \infty} 3 \cdot (1+0)^3 \cdot 0 = 0 < 1$$

Hence, by ratio test $\sum a_n$ convergent.

15) $\sum_{n=1}^{\infty} \frac{\sin(2n)}{2+2^n}$: let $a_n = \frac{\sin(2n)}{2+2^n}$

$$|a_n| = \frac{|\sin(2n)|}{2+2^n} \leq \frac{1}{2+2^n} < \frac{1}{2^n}.$$

Hence, by DCT with convergent geom. series $\sum \frac{1}{2^n}$ we have $\sum |a_n|$ is convergent.

So, $\sum a_n$ is absolutely convergent $\Rightarrow$ <u>convergent</u>.

16) $\sum_{n=1}^{\infty} \frac{1}{n+n!}$: let $a_n = \frac{1}{n+n!}$

Then,

$$a_n = \frac{1}{n+1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n} \leq \frac{1}{n+(n-1)\cdot n} = \frac{1}{n^2}$$

Hence, by DCT with convergent series $\sum \frac{1}{n^2}$ the series $\sum \frac{1}{n+n!}$ is <u>convergent</u>.

17) $\sum_{n=1}^{\infty} \frac{n}{n+n!}$; let $a_n = \frac{n}{n+n!}$

Then, $a_n = \frac{n}{n+n!}$

$$= \frac{n}{n+1\cdot 2\cdot \ldots (n-1)\cdot n}$$

$$= \frac{\cancel{n}}{\cancel{n}(1+1\cdot 2\cdot \ldots (n-1))}$$

$$= \frac{1}{1+(n-1)!} \leq \frac{1}{(n-1)!}$$

The series $\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{k=0}^{\infty} \frac{1}{k!}$ is

convergent: use ratio test: let $b_k = \frac{1}{k!}$

$$\left|\frac{b_{k+1}}{b_k}\right| = \frac{1}{(k+1)!} \cdot \frac{k!}{1} = \frac{\cancel{1}\cdot\cancel{2}\cdots\cancel{k}}{\cancel{1}\cdot\cancel{2}\cdots\cancel{k}(k+1)}$$

$$= \frac{1}{k+1} \xrightarrow{k\to\infty} 0 < 1$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$ is convergent and, by DCT the series $\sum a_n$ is convergent.

18) $\sum_{n=1}^{\infty} \frac{n^2}{n+n!}$: Let $a_n = \frac{n^2}{n+n!}$

Then, $a_n = \frac{n^2}{n+n!} = \frac{n}{1+(n-1)!}$

Try ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{1+n!} \cdot \frac{1+(n-1)!}{n}$$

$$= \frac{n+1}{n} \cdot \frac{1+1\cdot2\cdot3\cdots(n-1)\cdot\frac{n}{n}}{1+1\cdot2\cdot3\cdots(n-1)\cdot n}$$

$$= \left(1+\frac{1}{n}\right) \cdot \frac{\cancel{n!}}{\cancel{n!}} \cdot \left[\frac{\frac{1}{n!}+\frac{1}{n}}{\frac{1}{n!}+1}\right]$$

$$\xrightarrow{n\to\infty} (1+0)\cdot\left[\frac{0+0}{0+1}\right] = 0 < 1$$

Hence, by Ratio Test series converges.

19) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$: Let $a_k = \dfrac{2^k k!}{(k+2)!}$

Try ratio ~~test~~ :

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{k+1}(k+1)!}{(k+3)!} \cdot \frac{(k+2)!}{2^k k!}$$

$$= \frac{2^{k+1}}{2^k} \cdot \frac{\cancel{k!}\,(k+1)\cdot\cancel{(k+2)!}}{\cancel{k!}\,(k+3)\cdot\cancel{(k+2)!}}$$

$$= 2\left(\frac{k+1}{k+3}\right) = 2\,\frac{\cancel{k}}{\cancel{k}}\cdot\left(\frac{1+\frac{1}{k}}{1+3/k}\right)$$

$$\xrightarrow{k\to\infty} 2\cdot\left(\frac{1+0}{1+0}\right) = 2 > 1$$

Hence, by ratio test $\sum a_k$ diverges.

20) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+(-1)^n}$: Let $a_n = \dfrac{1}{2+(-1)^n}$

~~As~~ Note: $2+(-1)^n \leq 3$

$$\Rightarrow \frac{1}{3} \leq \frac{1}{2+(-1)^n}$$

Hence; not possible that $\lim a_n = 0$.

$\Rightarrow \sum a_n$ divergent, by Test for divergence.

21) See (11).

22) $\displaystyle\sum_{l=1}^{\infty} \frac{l^3+\sqrt{l}}{l^5+3l-1}$: Try LCT with $\sum \frac{1}{l^2}$.

Let $a_l = \dfrac{l^3+\sqrt{l}}{l^5+3l-1} > 0$, for all l. , $b_l = \dfrac{1}{l^2}$

$$\frac{a_l}{b_l} = \frac{l^3+\sqrt{l}}{l^5+3l-1} \cdot \frac{l^2}{1} = \frac{l^5+l^{5/2}}{l^5+3l-1}$$

$$= \frac{\cancel{l^5}\,(1+l^{-5/2})}{\cancel{l^5}\,(1+3l^{-4}-\frac{1}{l^5})}$$

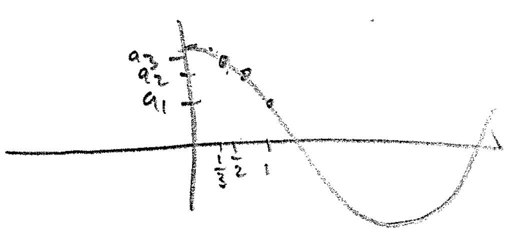
$$\xrightarrow{l\to\infty} \frac{1+0}{1+0-0} = 1 > 0.$$

Hence, since $\sum \frac{1}{l^2}$ converges (p-series, $p=2$) the same is true of $\sum a_l$, by LCT.

23) $\displaystyle\sum_{n=1}^{\infty} \cos(1/n)$: Let $a_n = \cos(1/n)$

As $n \to \infty$, $\cos(1/n) \longrightarrow 1 \neq 0$

By test for divergence, series is <u>divergent</u>.



24) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\sqrt{n}-1)}$: Let $a_n = \frac{1}{n(\sqrt{n}-1)}$ (11)

For $n=1,2,3,\ldots$, $a_n > 0$.

Apply LCT with $\sum \frac{1}{n^{3/2}}$:

Let $b_n = \frac{1}{n^{3/2}}$. Then,

$$\frac{a_n}{b_n} = \frac{1}{n(\sqrt{n}-1)} \cdot \frac{n^{3/2}}{1}$$

$$= \frac{1}{\cancel{n^{3/2}}\left(1-\frac{1}{\sqrt{n}}\right)} \cdot \frac{\cancel{n^{3/2}}}{1}$$

$$= \frac{1}{1-\frac{1}{\sqrt{n}}} \xrightarrow{n\to\infty} \frac{1}{1-0} = 1 > 0.$$

Hence, as $\sum \frac{1}{n^{3/2}}$ converges (p-series, $p = 3/2$) the same is true of $\sum a_n$, by LCT.