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May 9 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.1, 9.3.
- *Calculus II*, Marsden, Weinstein: Chapter 11.3.

Keywords: growth and decay equation, linear first-order equations

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Differential Equations

Separable equations

If a differential equation can be written in the form

$$\frac{dy}{dx} = g(x)h(y) \qquad (*)$$

where the right hand side factors as the product of a function of x and a function of y then it is called a **separable equation**.

We approach solving separable equations - i.e. determining all solutions $y = y(x)$ satisfying $(*)$ - as follows: rearranging $(*)$ we have

$$\frac{1}{h(y)} \cdot \frac{dy}{dx} = g(x)$$

then the method of inverse substitution gives

$$\int \frac{1}{h(y)} dy = \int g(x) dx + C$$

Example:

1. Consider the separable equation

$$\frac{dy}{dx} = 4y.$$

Here we can take $g(x) = 4$, $h(y) = y$. Then, we find

$$\_\_\_\_\_\_\_\_\_\_\_\_ = \int \frac{1}{y} dy = \int 4dx = \_\_\_\_\_\_\_\_\_\_\_\_$$

Thus,

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2. Consider the separable equation

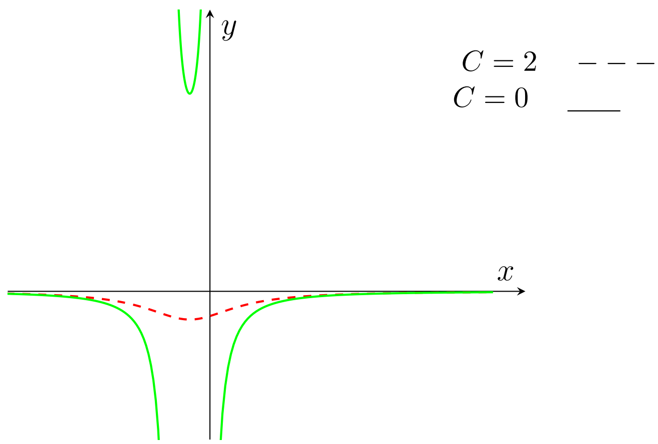
$$\frac{dy}{dx} = y^2(x+1)$$

Here $h(y) = y^2$ and $g(x) = x + 1$. Then, we find

$$\underline{\hspace{3cm}} = \int \frac{1}{y^2} dy = \int (x+1)dx = \underline{\hspace{3cm}}$$

so that

$$y = \underline{\hspace{3cm}}$$



Natural growth and decay

In this paragraph we consider an application of differential equations to problems involving growth and decay.

A separable differential equation of the form

$$\frac{dP}{dt} = kP, \qquad k \text{ constant}$$

is called a **Growth and Decay Equation** (GDE). A solution $P = P(t)$ describes growth/decay of a **population** $P(t)$ at time t.

Example:

1. A certain cell culture grows at an instantaneous rate of change proportional to the number of cells present at time t. If $P(t)$ denotes the number of cells at time t (i.e. the size of the population of cells at time t) then the previous sentence is formulated mathematically as an equation
$$\frac{dP}{dt} = kP$$

2. Radioactivity is a property characteristic of substances whose atoms undergo spontaneous decomposition or **decay**. The decay usually happens at some constant rate, releasing "bursts" that can be detected by a geiger counter. The more radioactive the sample, the more frequent the bursts, and the more intense the measured level of bursts. As the atoms decay, the rate of change of the mass of the radioactive isotope in the sample is proportional to the

mass present. (i.e. if there is twice as much radioactive material, then twice as many atoms would break apart in a given period of time).

Let $m(t)$ denote the mass of a radioactive substance at time t. Then, we have

$$\frac{dm}{dt} = -rm$$

for some positive constant $r > 0$. The negative sign is appearing because we require our substance mass to be **decreasing** - this means the instantaneous rate of change must be negative.

Using our method to solve separable differential equations, we can describe solutions to a GDE.

The general solution to the **Growth & Decay Equation**

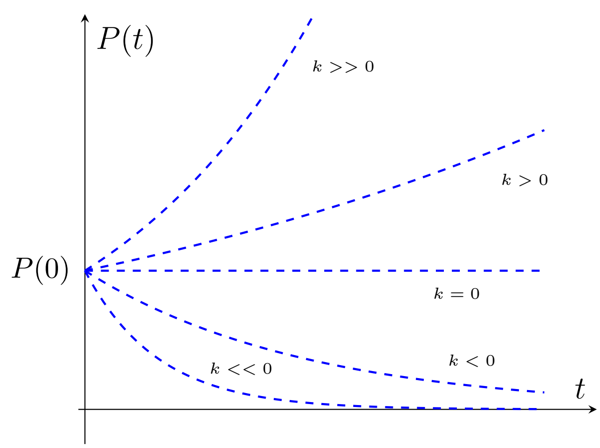
$$\frac{dP}{dt} = kP, \qquad k \text{ constant},$$

is given by

$$P(t) = \underline{\hspace{4cm}},$$

for some constant C. Moreover, given any real number C_0 there exists ____________________ solution satisfying $P(0) = C_0$.

The relationship between the sign of k and the behaviour of $P(t)$ is shown below.



Newton's Law of Cooling

Newton's Law of Cooling states the following:

The instantaneous rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature of its surroundings.

Let

$T(t) =$ temperature of object at time t.
$T(0) = T_0 =$ initial temperature of the object.
$T_a =$ ambient temperature of surrounding.

Newton's Law of Cooling states that there is a constant k such that

$$\frac{dT}{dt} = k(T - T_a).$$

Define

$y(t) = T(t) - T_a =$ temperature difference between object and surroundings at time t..

$y(0) = T_0 - T_a =$ initial temperature difference at $t = 0$.

Then, since T_a is a constant, we can rewrite the above differential equation as

$$\frac{dy}{dt} = ky$$

Hence, the rate at which an object cools relative to its surrounding is governed by a Growth & Decay Equation. In particular, its general solution is

$$y(t) = y_0 \exp(kt)$$

Hence, the temperature of the object at time t is

$$T(t) = y(t) + T_a = (T_0 - T_a)\exp(kt) + T_a$$

Linear First-Order Equations

A **linear first-order differential equation** is a differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (*)$$

where $P(x), Q(x)$ are continuous functions defined on some common interval.

Check your understanding

Which of the the following equations are linear first-order equations?

$$(A) :; \ \frac{dy}{dx} = xy^2 + x, \quad (B): \ \frac{dy}{dx} = g(x)h(y), \quad (C): \ \frac{dy}{dx} = ky, \quad (D): \ x^2y' + xy = 1$$

Let's consider how we may solve a linear first-order equation. We use the following **nifty trick**: we want to solve the general linear first-order equation $(*)$. Define

$$z = y \exp\left(\int P(x)dx\right)$$

Then, $z = z(x)$ and

$$\frac{dz}{dx} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Hence, we find

$$z = \underline{\hspace{5cm}}$$

$$\Rightarrow y = \underline{\hspace{7cm}}$$

In summary,

Method for Solving Linear First-Order Equations

(1) Compute an antiderivative of $P(x)$ i.e. determine $\int P(x)dx$ (no constant of integration required)

(2) Let $I(x) = \exp\left(\int P(x)dx\right)$. Multiply both sides of $(*)$ by $I(x)$.

(3) In this way $(*)$ becomes

$$\frac{d}{dx}\left(I(x)y\right) = Q(x)I(x) \qquad (***)$$

(4) Integrate both sides of $(***)$

(5) Rearrange for y.

Example: Solve

$$\frac{dy}{dx} + \frac{y}{x} = 1$$

for $x > 0$. Here

$$P(x) = \underline{\hspace{2.5cm}}, \qquad Q(x) = \underline{\hspace{2.5cm}}$$

and the integrating factor is

$$I(x) = \underline{\hspace{7cm}}$$

Then,

$$I(x)y = \int Q(x)I(x)dx = \underline{\hspace{4cm}}$$

$$\Longrightarrow \quad y = \underline{\hspace{4cm}}$$