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May 7 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 9.1, 9.3.
- *Calculus II*, Marsden, Weinstein: Chapter 11.3.

Keywords: differential equations, solutions,

Differential Equations

In this lecture we will begin an investigation of differential equations. These are quite different to the equations you have seen in the past in that solutions are *functions*, and not numbers. We discuss what differential equations are, what a solution to a differential equation is, and investigate a class of differential equations known as *separable equations*.

What is a differential equation?

Differential equations are equations involving the derivative or derivatives of an unknown function $f(x)$. For example,

$$\frac{df}{dx} = x\sin(x), \qquad (f'(x))^2 + f''(x) = x + 1, \qquad \frac{d^2 f}{dx^2} = -4f(x)$$

are all examples of differential equations.

> A **solution** of a differential equation is a function $f(x)$ whose derivatives satisfy the given equation.

For example, the function $f(x) = \cos(2x)$ is a solution to the differential equation

$$f''(x) = -4f(x)$$

Indeed, we check that

$$f''(x) = -4\cos(4x) = -4f(x).$$

> **Aim:** given a differential equation, find all solutions $f(x)$.

Remark: Recall the notation:

$$\frac{d}{dx}f(x) = \frac{df}{dx} = f'(x)$$

Example:

1. To **solve** the differential equation

$$\frac{df}{dx} = x\sin(x)$$

we seek *all* functions $f(x)$ which satisfy

$$\frac{d}{dx}f(x) = x\sin(x), \qquad \text{or, equivalently,} \qquad f'(x) = x\sin(x)$$

CHECK YOUR UNDERSTANDING

(a) Verify that $f(x) = -x\cos(x) + \sin(x) - 5$ is a solution to the above differential equation.

(b) Express the differential equation as an antiderivative problem to complete the following statement:

Find $f(x)$ such that $f(x) =$ ____________________

(c) Find all possible solutions to the differential equation

$$\frac{d}{dx}f(x) = x\sin(x)$$

2. Consider the differential equation

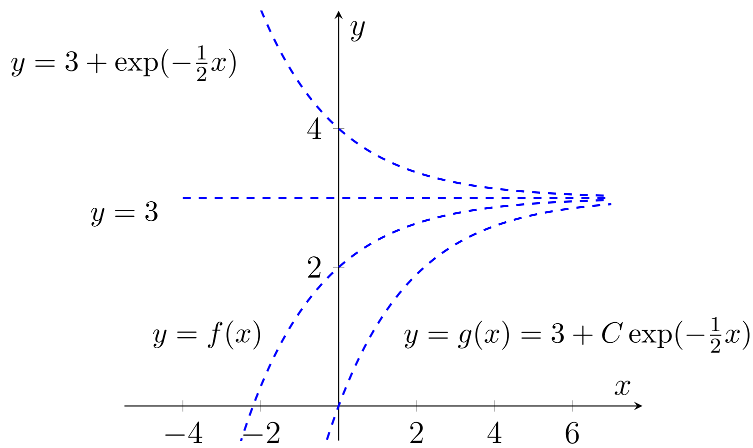
$$\frac{df}{dx} = \frac{3}{2} - \frac{1}{2}f(x)$$

CHECK YOUR UNDERSTANDING

(a) Which of the following functions $f(x)$ are solutions to the differential equation?

- $f(x) = \frac{3}{2}x - \frac{1}{2}x^2$
- $f(x) = 3$
- $f(x) = 3 - 5\exp(-\frac{1}{2}x)$
- $f(x) = 3 + C\exp(-\frac{1}{2}x)$, for any constant C

(b) The graphs of four solutions to the differential equation are given below:



Using the graph, determine a such that the solution $f(x)$ satisfies $f(0) = a$

Using the graph, determine C such $g(0) = 0$.

Separable equations

If a differential equation can be written in the form

$$\frac{dy}{dx} = g(x)h(y) \qquad (*)$$

where the right hand side factors as the product of a function of x and a function of y then it is called a **separable equation**.

Example: The following differential equations are separable:

$$(A): \ \frac{dy}{dx} = \frac{-3x}{y + yx^2}, \quad (B): \ \frac{dP}{dt} = kP, \quad (C): \ yy' = \cos(2x)$$

We approach solving separable equations - i.e. determining all solutions $y = y(x)$ satisfying $(*)$ - as follows: rearranging $(*)$ we have

$$\frac{1}{h(y)} \cdot \frac{dy}{dx} = g(x)$$

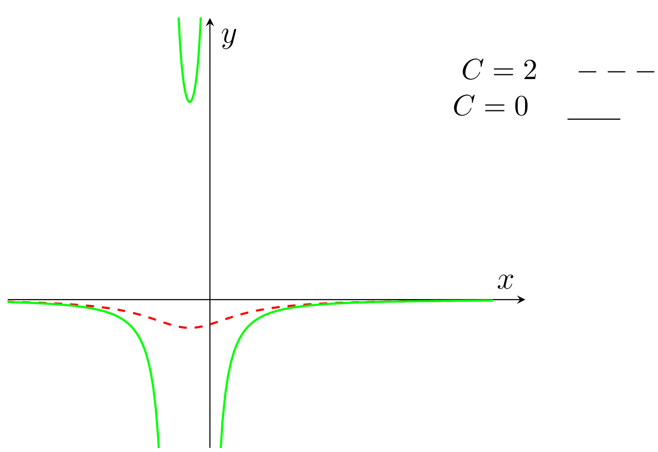
then the method of inverse substitution gives

$$\int \frac{1}{h(y)} dy = \int g(x) dx + C$$

Example:

1. Consider the separable equation
$$\frac{dy}{dx} = 4y.$$
Here we can take $g(x) = 4$, $h(y) = y$. Then, we find
$$\underline{\hspace{3cm}} = \int \frac{1}{y} dy = \int 4 dx = \underline{\hspace{3cm}}$$
Thus,
$$\underline{\hspace{6cm}}$$

2. Consider the separable equation
$$\frac{dy}{dx} = y^2(x+1)$$
Here $h(y) = y^2$ and $g(x) = x + 1$. Then, we find
$$\underline{\hspace{3cm}} = \int \frac{1}{y^2} dy = \int (x+1) dx = \underline{\hspace{3cm}}$$
so that
$$y = \underline{\hspace{3cm}}$$



Check your understanding

Write down a separable differential equation that describes a curve whose slope at (x, y) is xy.

Find an equation of the curve passing through the point $(1, 0)$ and whose slope at (x, y) is xy.