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May 4 Lecture

SUPPLEMENTARY REFERENCES:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.8.
- *Calculus II*, Marsden, Weinstein: Chapter 11.3.
- *AP Calculus BC*, Khan Academy: Improper Integrals.

KEYWORDS: improper integrals, the Integral Test

Improper Integrals

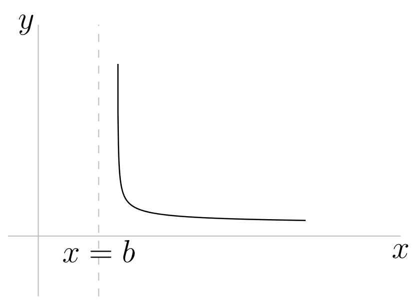
Type II Improper Integrals

We will consider how to approach determining the area under the graph of a a function that admits infinite discontinuities.

RECALL: Let $f(x)$ be a nonnegative function, continuous on $[a, b)$ or $(b, a]$ and suppose $\lim_{x\to b} f(x) = +\infty$. Then, $x = b$ is called an **infinite discontinuity of** $f(x)$.

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES!

Consider the function $f(x) = \frac{1}{\sqrt{x-2}}$. A portion of the graph of $f(x)$ is shown below



1. Determine b so that $f(x)$ admits an infinite discontinuity at $x = b$.

2. Let a be a real number so that $b < a < 5$. Determine
$$\int_a^5 \frac{1}{\sqrt{x-2}}dx$$

3. Is the area between the graph of $f(x)$ and the x-axis finite or infinite? If finite, what is the area? If infinite, explain why.

The above investigation leads us to the following definition.

Type II Improper Integrals

Let $f(x)$ be a nonnegative function. Suppose that $x = b$ is an infinite discontinuity of $f(x)$.

- Suppose $f(x)$ is continuous on $[a, b)$. If $\lim_{t\to b} \int_a^t f(x)dx$ exists (and is finite) then we define

$$\int_a^b f(x)dx \stackrel{def}{=} \lim_{t\to b} \int_a^t f(x)dx$$

- Suppose $f(x)$ is continuous on $(b, a]$. If $\lim_{t\to b} \int_t^a f(x)dx$ exists (and is finite) then we define

$$\int_b^a f(x)dx \stackrel{def}{=} \lim_{t\to b} \int_t^b f(x)dx$$

In either case, we say that $\int_a^\infty f(x)dx$ (resp. $\int_{-\infty}^b f(x)dx$) is a **convergent (improper) integral**. Otherwise, the (improper) integral is **divergent**.

Remark: An integral $\int_a^b f(x)dx$ defined over an interval $[a, b]$ on which $f(x)$ admits an infinite discontinuity is called a **type II improper integral**. It is not an integral in the usual sense (i.e. it is not defined as the limit of Riemann sums).

Example:

1. Consider the improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$$

Since the integrand $\frac{1}{\sqrt{1-x^2}}$ admits an infinite discontinuity at __________ the integral is a type II improper integral. Hence, by definition

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \underline{\hspace{8cm}}$$

$$= \underline{\hspace{9cm}}$$

Hence, the improper integral is _______________.

2. Consider the function $f(x) = \frac{1}{x-2}$. There exists an infinite discontinuity of $f(x)$ at $x = 2$. Then, the integral

$$\int_0^2 \frac{1}{x-2}dx$$

is improper. Moreover, by definition

$$\int_2^5 \frac{1}{x-2}dx = \lim_{t\to 2}\int_2^5 \frac{1}{x-2}dx = \lim_{t\to 2}\left[\log(x-2)\right]_t^5 = \lim_{t\to 2}\left(\log(3)-\log(t-2)\right) = +\infty$$

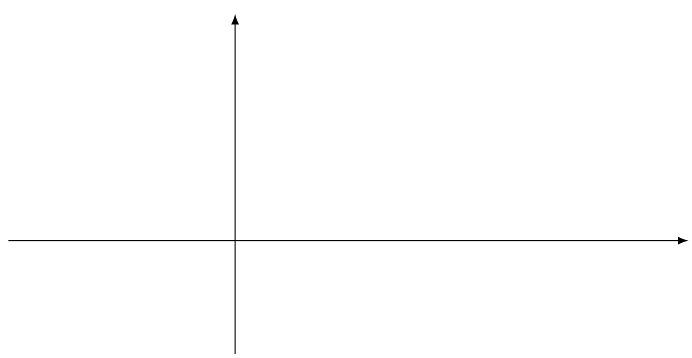
Hence, the improper integral is divergent.

The Integral Test for Series

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms, $a_n \geq 0$. Suppose that $a_n = f(n)$, for a continuous function $f(x)$ defined for $1 \leq x < \infty$. Furthermore, assume that $f(x)$ is a **monotone decreasing function**: this means that if $x \leq y$ then $f(x) \geq f(y)$.

Check your understanding

Draw the general shape of the graph of a monotone decreasing function.

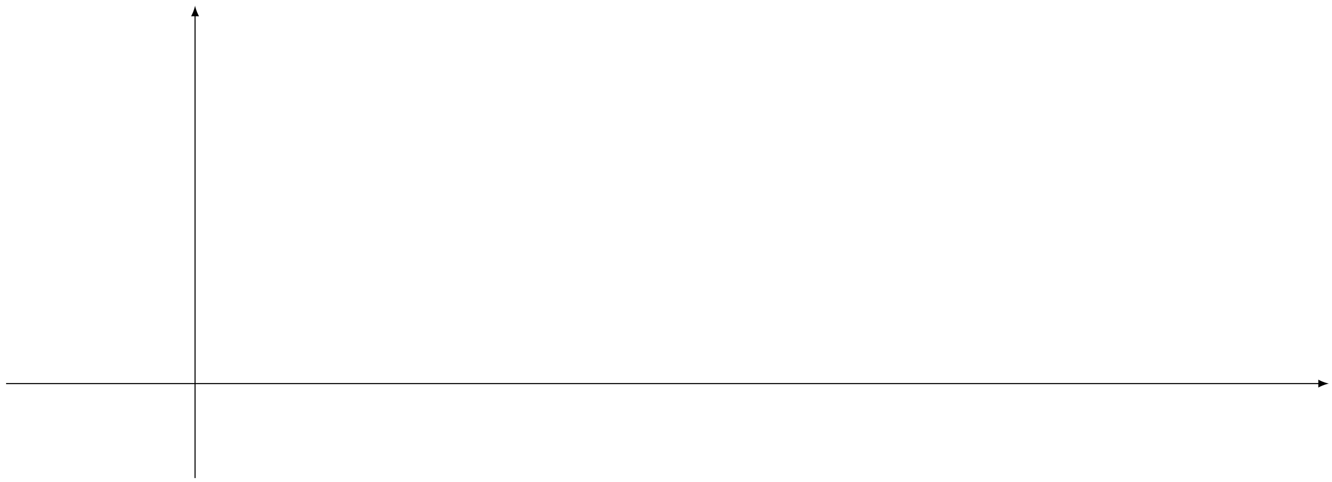


For example, the p-series, $p > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is such a series.

We can use the graph of $f(x)$ to **visualise** the series $\sum_{n=1}^{\infty} a_n$:



In particular,

$\sum_{n=2}^{\infty} a_n$ ________________ whenever ______________________

This leads to the following new test for convergence of series.

Integral Convergence Test for Series

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that $a_n = f(n)$, where $f(x)$ is a positive, continuous, monotone decreasing function defined on $1 \leq x < \infty$.

- $\sum_{n=1}^{\infty} a_n$ converges whenever

  ____

  is ________________

- $\sum_{n=1}^{\infty} a_n$ diverges whenever

  ____

  is ________________

Example: We can use the integral test to give a quick and easy proof of the divergence of the Harmonic Series. Let $f(x) = \frac{1}{x}$. We saw in the last lecture that

$$\int_1^{\infty} \frac{1}{x} dx$$

is divergent. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.