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May 3 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.8.
- *Calculus II*, Marsden, Weinstein: Chapter 11.3.
- *AP Calculus BC*, Khan Academy: Improper Integrals.

Keywords: improper integrals

Improper Integrals

In this lecture we will investigate what we could mean by an integral on an *unbounded domain*. We will define *improper integrals of (type I and type II)* and give some examples.

Unbounded definite integrals

Given any continuous function $f(x)$, defined on the closed interval $[a, b]$, we *define*

$$\int_a^b f(x)dx \quad = \quad \text{(signed) area lying between graph } y = f(x) \text{ and } x\text{-axis, for } a \le x \le b$$

In particular, definite integrals have only been defined on **bounded intervals**.

> Question: how might we define definite integrals on **unbounded intervals**?

Remark: We can't mimic the bounded interval case and use Riemann sums: this requires us to subdivide an interval $[a, b]$ into n subintervals and consider the limit of sums of areas of rectangles. In the unbounded case we would necessarily be forced to consider rectangles having 'infinite' base!

Mathematical workout - flex those muscles

Consider the function $f(x) = \frac{1}{x^2}$. For a natural number n define

$$a_n = \int_1^n f(x)dx$$

1. Evaluate a_1, a_2, a_{10}.

2. SPOT THE PATTERN! Determine the general expression:

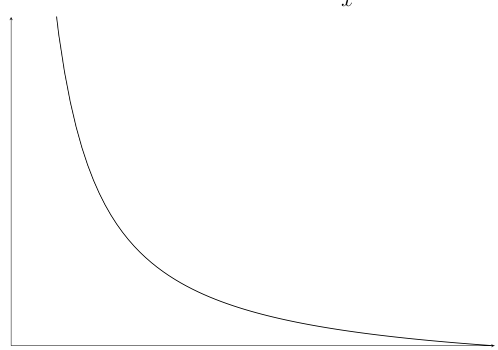
$a_n =$ ______________

3. Circle the correct description.

- (a_n) is BOUNDED UNBOUNDED
- (a_n) is MONOTONIC NON-MONOTONIC

4. Explain your second choice above using the graph.

$y = \frac{1}{x^2}$



5. Is the sequence (a_n) convergent? If so, determine $L = \lim_{n\to\infty} a_n$. If not, justify.

6. Is the area under the (infinite) graph $y = \frac{1}{x^2}$, where $x \geq 1$, finite or infinite? If finite, determine this area. If infinite, explain why.

1 Type I Improper Integrals

- Let $f(x)$ be a nonnegative function, continuous on the interval $[a, \infty)$. If $\lim_{t\to\infty} \int_a^t f(x)dx$ exists (and is finite) then we define

$$\int_a^\infty f(x)dx \stackrel{def}{=} \lim_{t\to\infty} \int_a^t f(x)dx$$

- Let $f(x)$ be a nonnegative function, continuous on the interval $(-\infty, b]$. If $\lim_{t\to-\infty} \int_t^b f(x)dx$ exists (and is finite) then we define

$$\int_{-\infty}^b f(x)dx \stackrel{def}{=} \lim_{t\to-\infty} \int_t^b f(x)dx$$

In either case, we say that $\int_a^\infty f(x)dx$ (resp. $\int_{-\infty}^b f(x)dx$) is a **convergent (improper) integral**. Otherwise, the (improper) integral is **divergent**.

- An (improper) integral

$$\int_{-\infty}^\infty f(x)dx \stackrel{def}{=} \int_0^\infty f(x)dx) + \int_{-\infty}^0 f(x)dx$$

is **convergent** if and only if *both the integrals* on the right hand side are convergent. In particular, the improper integral is **divergent** if either of the improper integrals on the right hand side are divergent.

Remark:

1. An integral (irrespective of convergence) of the form

$$\int_a^\infty f(x)dx \qquad \text{or} \qquad \int_{-\infty}^b f(x)dx \qquad \text{or} \qquad \int_{-\infty}^\infty f(x)dx$$

   is called an **improper integral**. Improper integrals are not 'proper' because they are not obtained as a limit of Riemann sums. This is important to remember.

2. Improper integrals can also be defined for arbitrary (i.e. not necessarily nonnegative) functions. However, we will only consider the nonegative situation.

Example:

1. Consider the improper integral

$$\int_1^\infty \frac{1}{x}dx$$

   By definition,

$$\int_1^\infty \frac{1}{x}dx = \underline{\hspace{4cm}}$$

We have

$$\int_1^t \frac{1}{x} dx = \underline{\hspace{6cm}}$$

Since $\ln(t)$ is $\underline{\hspace{2.5cm}}$ as $t \to \infty$, the improper integral $\int_1^\infty \frac{1}{x} dx$ is $\underline{\hspace{2.5cm}}$.

2. Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$$

The integrand is unchanged under the change of variable $x \leftrightarrow -x$, so that

$$\int_0^a \frac{1}{x^2+1} dx = \underline{\hspace{3cm}}, \qquad \text{for any } a.$$

Hence, $\int_0^\infty \frac{1}{x^2+1} dx$ is convergent if and only if $\int_{-\infty}^0 \frac{1}{x^2+1} dx$ is convergent. If either (hence, both) of these integrals are convergent then

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \underline{\hspace{3cm}}$$

We compute

$$\int_0^a \frac{1}{x^2+1} dx = [\arctan(x)]_0^a = \arctan(a)$$

Now,

$$t = \arctan(a) \quad \Leftrightarrow \quad \tan(t) = a$$

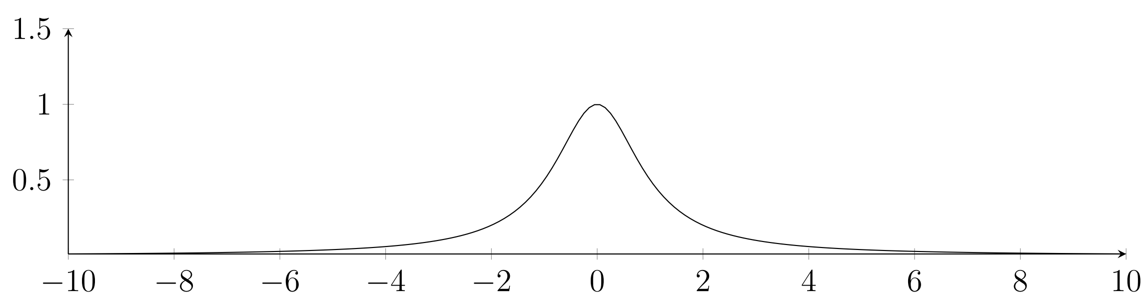
Hence, as $a \to \infty$,

$$t = \arctan(a) \longrightarrow \underline{\hspace{1.5cm}}$$

Hence, the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$$

is $\underline{\hspace{2.5cm}}$.



Check your understanding

Determine the convergence of the improper integral

$$\int_0^\infty \exp(-2x)dx$$

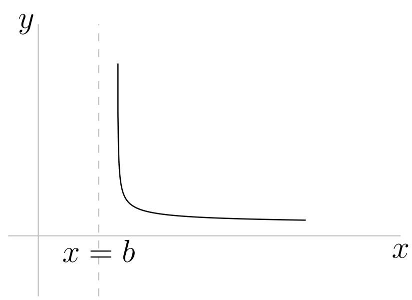
Type II Improper Integrals

We will consider how to approach determining the area under the graph of a a function that admits infinite discontinuities.

Recall: Let $f(x)$ be a nonnegative function, continuous on $[a,b)$ or $(b,a]$ and suppose $\lim_{x\to b} f(x) = +\infty$. Then, $x = b$ is called an **infinite discontinuity of** $f(x)$.

Mathematical workout - flex those muscles!

Consider the function $f(x) = \frac{1}{\sqrt{x-2}}$. A portion of the graph of $f(x)$ is shown below



1. Determine b so that $f(x)$ admits an infinite discontinuity at $x = b$.

2. Let a be a real number so that $b < a < 5$. Determine

$$\int_a^5 \frac{1}{\sqrt{x-2}}dx$$

3. Is the area between the graph of $f(x)$ and the x-axis finite or infinite? If finite, what is the area? If infinite, explain why.

The above investigation leads us to the following definition.

Type II Improper Integrals

Let $f(x)$ be a nonnegative function. Suppose that $x = b$ is an infinite discontinuity of $f(x)$.

- Suppose $f(x)$ is continuous on $[a, b)$. If $\lim_{t\to b} \int_a^t f(x)dx$ exists (and is finite) then we define

$$\int_a^b f(x)dx \stackrel{def}{=} \lim_{t\to b} \int_a^t f(x)dx$$

- Suppose $f(x)$ is continuous on $(b, a]$. If $\lim_{t\to b} \int_t^a f(x)dx$ exists (and is finite) then we define

$$\int_b^a f(x)dx \stackrel{def}{=} \lim_{t\to b} \int_t^b f(x)dx$$

In either case, we say that $\int_a^\infty f(x)dx$ (resp. $\int_{-\infty}^b f(x)dx$) is a **convergent (improper) integral**. Otherwise, the (improper) integral is **divergent**.

Remark: An integral $\int_a^b f(x)dx$ defined over an interval $[a, b]$ on which $f(x)$ admits an infinite discontinuity is called a **type II improper integral**. It is not an integral in the usual sense (i.e. it is not defined as the limit of Riemann sums).

Example:

1. Consider the improper integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$$

Since the integrand $\frac{1}{\sqrt{1-x^2}}$ admits an infinite discontinuity at $x = 1$ the integral is a type II improper integral. Hence, by definition

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to 1} \int_0^t \frac{1}{\sqrt{1-x^2}}dx = \lim_{t\to 1} \arcsin(t) = \frac{\pi}{2}.$$

Hence, the improper integral is convergent.

2. Consider the function $f(x) = \frac{1}{x-2}$. There exists an infinite discontinuity of $f(x)$ at $x = 2$. Then, the integral

$$\int_0^2 \frac{1}{x-2}dx$$

is improper. Moreover, by definition

$$\int_2^5 \frac{1}{x-2}dx = \lim_{t\to 2} \int_2^5 \frac{1}{x-2}dx = \lim_{t\to 2} \left[\log(x-2)\right]_t^5 = \lim_{t\to 2} \left(\log(3) - \log(t-2)\right) = +\infty$$

Hence, the improper integral is divergent.