March 9 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.5.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Ratio & alternating series tests.

Keywords: conditional covergence, absolute convergence

Absolute & conditional convergence

Recall: the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent while the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Let $\sum a_n$ be a series. If the series $\sum |a_n|$ is convergent then we say that the original series $\sum a_n$ is **absolutely convergent.** If a series $\sum a_n$ is convergent but not absolutely convergent then we say that $\sum a_n$ is **conditionally convergent.**

Check your understanding

Which of the following series are absolutely convergent, conditionally convergent, neither.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

$\left|\frac{(-1)^{n-1}}{n^2}\right| = \frac{1}{n^2}$, $\sum \frac{1}{n^2}$ convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ absolutely convergent.

2. $\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n+3^n}$

NEITHER

Since $\lim \frac{(-3)^n}{2^n+3^n}$ does not exist as $n \to \infty$, series not conditionally convergent

Similarly, $\lim \frac{3^n}{2^n+3^n} = 1 \neq 0 \Rightarrow$ not absolutely convergent.

3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$a_n = (-1)^{n-1} \frac{1}{\sqrt{n+1}+\sqrt{n}}$; $|a_n| = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and $\sum |a_n|$ divergent, by LCT with $\sum \frac{1}{\sqrt{n}}$.

Series is convergent: by AST $b_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$;
- $\lim b_n = 0$
- $b_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{\sqrt{n+1}+\sqrt{n+2}} = b_{n+1}$

Hence, (b_n) decreasing

Hence, conditionally convergent.

Absolute convergence has the following useful consequence.

Absolute convergence implies convergence

If a series $\sum a_n$ is absolutely convergent then it is convergent.

Proof: If $\sum a_n$ is absolutely convergent then $\sum |a_n|$ is convergent and the same is true of the series $\sum 2|a_n|$.

Observation: For any real number x, $0 \le x + |x| \le 2|x|$, (since $|x|$ is either x or $-x$).

Hence, applying the DCT we see that $\sum(a_n + |a_n|)$ is convergent. Now,

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

is a difference of two convergent series, and therefore convergent.

Example:

1. Consider the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$. Then,

$$\left|\frac{\sin(n)}{n^2}\right| \le \frac{1}{n^2}.$$

   Hence, by the DCT the series $\sum \left|\frac{\sin(n)}{n^2}\right|$ is convergent. Thus, the series $\sum \frac{\sin(n)}{n^2}$ is absolutely convergent, hence convergent.

2. Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+3}{2n^3+5n-1}$. Then,

$$\left|(-1)^{n-1}\frac{n+3}{2n^3+5n-1}\right| = \frac{n+3}{2n^3+5n-1}$$

   The series $\sum_{n=1}^{\infty} \frac{n+3}{2n^3+5n-1}$ is convergent by the LCT. Hence, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+3}{2n^3+5n-1}$ is absolutely convergent, hence convergent.

Warning! Conditionally convergent series provide demonstrations of some of the weird things that can happen with series if we consider them as *infinite sums* (which they are not). For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is conditionally convergent with limit L. Now, suppose that we consider this series as an *infinite sum*, and write

$$L = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \cdots \qquad \text{(A)}$$

Then,

$$\frac{L}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots$$

whe we can rewrite as

$$\frac{L}{2} = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 \cdots \qquad \text{(B)}$$

Now, we add $(A)+(B)$

$$L = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots \tag{A}$$

$$\frac{L}{2} = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} \tag{B}$$

to get

$$\frac{3L}{2} = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} \ldots$$

It's not too difficult to show that this last infinite series contains the same terms as (A), but **rearranged** so that one negative term occurs after two positive terms. Hence, $L = \frac{3L}{2} \implies L = 0$, which contradicts the fact that $\frac{1}{2} = s_2 \leq L \leq s_1 = 1$...!

The problem here is the process of **rearrangement**: for finite sums we are free to rearrange terms however we please (in fancy algebraic language, *addition is commutative*). However, as we've just demonstrated, we must be careful when attempting to rearrange the terms of a (infinite) series.

The situation for absolutely convergent series is much more straightforward:

> Let $\sum a_n$ be an absolutely convergent series. If (b_n) is a rearrangement of the terms of the sequence (a_n) (so that (b_n) has the terms as (a_n) but listed in a different order) then $\sum b_n = \sum a_n$.

Remark: Bernhard Riemann (1820-1866), one of history's most celebrated mathematicians, proved the following remarkable result.

> **Riemann Rearrangement Theorem**
>
> Let $\sum a_n$ be an ~~absolutely~~ conditionally convergent series. If (b_n) is a rearrangement of the terms of the sequence (a_n) (so that (b_n) has the terms as (a_n) but listed in a different order) then $\sum b_n = \sum a_n$.

For example, this Theorem states that there is a rearrangement (b_n) of

$$\left(\frac{(-1)^{n+1}}{n}\right) = \left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right)$$

so that

$$\sum_{n=1}^{\infty} b_n = 10^{10^{10^{10^{10^{10}}}}}$$