Middlebury College

Calculus II: Spring 2018

Contact: `gmelvin@middlebury.edu`

MARCH 5 LECTURE

SUPPLEMENTARY REFERENCES:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.5.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Ratio & alternating series tests.

KEYWORDS: alternating series, Alternating Series Test.

SERIES CONVERGENCE TESTS IV

Alternating series

We have considered tests of convergence for series having *positive terms* e.g. Direct Comparison Tests and Limit Comparison Tests. This doesn't help us out when we are interested in determining convergence of series whose terms are not positive. Let's investigate the progress that we can make.

Definition: A series of the form $\sum(-1)^n b_n$, where $b_n > 0$ for all n, is called an **alternating series**.

Remark:

1. An alternating series is a series whose successive terms have alternating sign.
2. A *necessary condition* that an alternating series is convergent is that $\lim_{n\to\infty} b_n = 0$ (this is true of any convergent series). However, this condition is *not sufficient*, as we will soon see.

Example: The following series are examples of alternating series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}, \qquad \sum_{n=3}^{\infty} \frac{n\cos(n\pi)}{n^2+4n+4},$$

Here we use that $\cos(n\pi) = (-1)^n$. The following series are *not* alternating:

$$\sum_{n=1}^{\infty} \frac{2-(-1)^n}{n}, \qquad \sum_{n=4}^{\infty} \frac{\sin(n)}{n^2}.$$

To check whether a series $\sum_n a_n$ is alternating you must show

$$a_{n+1} = -a_n, \quad \text{for every } n = 1, 2, 3, \ldots$$

We would like to determine conditions that the sequence (b_n) must satisfy so that the alternating series $\sum(-1)^n b_n$ is convergent. We already know that if $\sum(-1)^n b_n$ is convergent then it must be the case that $\lim_{n\to\infty} b_n = 0$. Do we require any further conditions?

Check your understanding

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots$$

This is an alternating series with $b_n = \frac{1}{2n+1}$, $n = 1, 2, 3, \dots$.

1. What adjectives would you use to describe the sequence (b_n)?

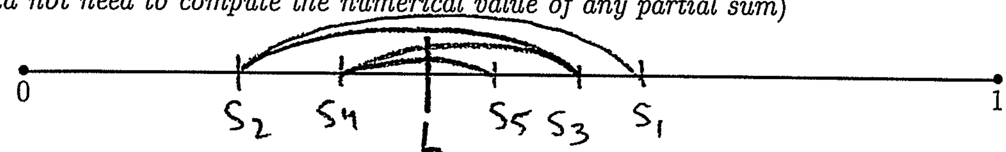
   decreasing, bounded, convergent

2. Write down the first five partial sums s_1, s_2, s_3, s_4, s_5. You do not need to simplify your expressions.

   $S_1 = \frac{1}{3}$ $\quad S_3 = \frac{29}{105} \quad S_5 = \frac{2661}{10395}$

   $S_2 = \frac{2}{15} \quad S_4 = \frac{156}{945}$

3. On the number line above plot a rough estimate of the values s_1, s_2, s_3, s_4, s_5. (*Hint: you should not need to compute the numerical value of any partial sum*)



4. Do you think the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ is convergent or divergent? If you think the series is convergent, indicate on the number line where its limit L will be; if you think the series is divergent, justify your conclusion.

   Convergent

5. Complete the following statement:

Alternating Series Test (AST) Let $\sum_n (-1)^n b_n$ be an alternating series, where $b_n > 0$ for $n = 1, 2, 3, \dots$. Suppose (b_n) is decreasing and $\lim_{n\to\infty} b_n = 0$ Then, the series $\sum_n (-1)^n b_n$ is convergent.

Example:

1. Consider the alternating series given above:

$$(a)\ \sum_{n=0}^{\infty}\frac{(-1)^n}{n},\qquad (b)\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1},\qquad (c)\ \sum_{n=3}^{\infty}\frac{n\cos(n\pi)}{n^2+4n+4},$$

Then,

(a) convergent: $b_n = \frac{1}{n}$ is decreasing and $\lim_{n\to\infty} b_n = 0$. Hence, by AST the series is convergent.

(b) divergent: let $a_n = (-1)^{n-1}\frac{n}{n+1} = (-1)^{n-1}\frac{1}{1+\frac{1}{n}}$. Then, (a_n) is not convergent so that $\lim a_n$ does not exist. Hence, the series is divergent by the Divergence Test.

(c) convergent: observe that $\cos(n\pi) = (-1)^n$. This series is an alternating series with $b_n = \frac{n}{n^2+4n+4} = \frac{n}{(n+2)^2}$. Using Limit Laws we can show $\lim_{n\to\infty} b_n = 0$. To show that b_n is decreasing we proceed as follows: let $f(x) = \frac{x}{(x+2)^2}$. Then, recalling the quotient rule for differentiation, we find

$$f'(x) = \frac{(x+2)^2 - 2x(x+2)}{(x+2)^4} = \frac{-(2x^2-x-2)}{(x+2)^3}$$

Since $2x^2 - x - 2 > 0$ whenever $x > 1$, we have $f'(x) < 0$ whenever $x > 1$. Hence, $f(x)$ is decreasing and the same is true of $b_n = f(n)$, $n \geq 3$.
Therefore, the series is convergent by the AST.

CHECK YOUR UNDERSTANDING

Determine whether the following alternating series are convergent or divergent.

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n!}$ (*Recall that* $n! = n\cdot(n-1)\cdot\dots\cdot 2\cdot 1$)

$b_n = \frac{2}{n!}$; $b_n - b_{n+1} = \frac{2}{n!} - \frac{2}{(n+1)!}$

$= \frac{2}{n!}\left(1 - \frac{1}{n+1}\right) > 0$

Hence, $b_n > b_{n+1}$ and (b_n) decreasing

& $\lim \frac{2}{n!} = 0$: $0 < \frac{1}{n!} \leq \frac{1}{n}$ and $\lim \frac{1}{n} = 0 \Rightarrow$ By Squeeze Thm,

$\lim \frac{1}{n!} = 0$.

Hence, $\sum(-1)^{n-1}\frac{2}{n!}$ convergent by AST.

2. $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$

- $b_n = \frac{n}{n^2+1} = \frac{n}{n^2}\cdot\frac{1}{1+1/n^2}$

$= \frac{1}{n}\cdot\frac{1}{1+1/n^2}$

As $n\to\infty$, $b_n \to 0\cdot 1$, by Limit Laws

- To show decreasing: $f(x) = \frac{x}{x^2+1}$

$f'(x) = \frac{(1+x^2) - 2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \leq 0$ when $x \geq 1$

Hence, (b_n) decreasing $\Rightarrow \sum(-1)^n\frac{n}{n^2+1}$ convergent by AST

3. $\sum_{n=1}^{\infty}(-1)^n \frac{2n^2}{n^2+n+1}$ Divergent by Test for divergence

as $n \to \infty$, $\frac{2n^2}{n^2+n+1} \longrightarrow 2$, by Limit Laws

$\Rightarrow a_n = (-1)^n \frac{2n^2}{n^2+n+1}$ non-convergent

$\Rightarrow$ Series diverges.

Caution!

We give an example of a divergent alternating series $\sum_n (-1)^n b_n$ where $\lim_{n\to\infty} b_n = 0$ but (b_n) is **not decreasing.**

Consider the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n, \quad \text{where} \quad b_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd,} \\ \frac{1}{2^n}, & \text{if } n \text{ even.} \end{cases}$$

Observe that the first few terms of the sequence (b_n) are

$$1, \frac{1}{4}, \frac{1}{3}, \frac{1}{16}, \frac{1}{5}, \frac{1}{64}, \ldots$$

In particular, the sequence is not decreasing. The partial sums are

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 - \tfrac{1}{4} \\ s_3 &= 1 - \tfrac{1}{4} + \tfrac{1}{3} \\ s_4 &= 1 - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{16} \\ &\vdots \end{aligned}$$

It can be shown that if $m = 2k$ is a very large even integer then

$$s_{2k} = 1 + \frac{1}{3} + \ldots + \frac{1}{2k-1} - \frac{1}{4} \cdot \frac{1-(-\frac{1}{4})^k}{1+\frac{1}{4}}$$

For k very large (hence m very large), it can be shown that

$$s_m = s_{2k} > \frac{1}{2}H_k - 1,$$

where (H_k) is the sequence of partial sums associated to the Harmonic Series $\sum_{n=1}^{\infty}\frac{1}{n}$. As the sequence (H_k) is unbounded, the same is true of the sequence (s_m). Hence, the series $\sum(-1)^{n+1}b_n$ is divergent.