Middlebury College

Calculus II: Spring 2018

Contact: `gmelvin@middlebury.edu`

MARCH 2 LECTURE

SUPPLEMENTARY REFERENCES:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.4.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Comparison Tests.

KEYWORDS: *Limit Comparison Test.*

CONVERGENCE TESTS FOR SERIES IV

Today we consider the **Limit Comparison Test** for series. This is a test for convergence of a series of positive terms.

At the end of yesterday we saw that we couldn't apply the Direct Comparison Test and compare the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

with the convergent geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ '*looks like*' the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and it seems entirely reasonable to expect that the convergence behaviour of these series should coincide. In order to show this, we had to use the following *magic formula* (which we can't yet prove is true)

$$\frac{1}{2^n - 1} \leq \frac{3}{2^n}, \quad \text{for } n = 1, 2, 3, \ldots$$

Then, since the series $\sum_{n=1}^{\infty} \frac{3}{2^n}$ is convergent (Geometric Series Theorem), we can apply the Direct Comparison Test to deduce that $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent.

The following test for convergence formalises this notion of comparing series with unknown convergence behaviour with *similar looking* convergent series.

Limit Comparison Test (LCT)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If the sequence $\left(\frac{a_n}{b_n}\right)$ is convergent and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge or both series diverge.

Example:

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$. As n gets very large the terms of the series begin to *look like* $\frac{1}{\sqrt{n}}$ (i.e. for n very large, $\sqrt{n+2}$ is approximately $\sqrt{n}$).

   The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2}$, and therefore divergent. It seems reasonable to expect that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is also divergent. However, since

   $$\frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n}}, \quad \text{for } n = 1, 2, 3, \ldots$$

   we can't apply the DCT to show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is divergent.

   However, if we let $b_n = \frac{1}{\sqrt{n+2}}$ and $a_n = \frac{1}{\sqrt{n}}$ then

   $$\frac{a_n}{b_n} = \frac{\sqrt{n+2}}{\sqrt{n}} = \frac{\sqrt{n(1+2/n)}}{\sqrt{n}} = \sqrt{1+2/n} \longrightarrow \sqrt{1+0} = 1, \quad \text{as } n \to \infty$$

   Hence, since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (p-series with $p = \frac{1}{2}$), the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is also divergent, by the Limit Comparison Test.

2. Let $a_n = \frac{1}{2^n}$, $b_n = \frac{1}{2^n - 1}$. Then,

   $$\frac{a_n}{b_n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \longrightarrow 1 - 0 = 1, \quad \text{as } n \to \infty.$$

   Hence, since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent (Geometric Series Theorem), the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is also convergent, by the Limit Comparison Test.

FLEX THOSE MATHEMATICAL MUSCLES!

Use the Limit Comparison Test to determine whether the following series converge or diverge. You will need to come up with a series of postive terms whose convergence behaviour you know to compare

1. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$ "looks like" (as $n \to \infty$) $\sum_{n=1}^{\infty} \frac{1}{n}$

Let $a_n = \frac{1}{\sqrt{n^2+1}}$

$b_n = \frac{1}{n}$

Then, $\frac{a_n}{b_n} = \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = \frac{n}{\sqrt{n^2+1}}$

$= \frac{n}{\sqrt{n^2(1+1/n^2)}} = \frac{\cancel{n}}{\cancel{n}} \cdot \frac{1}{\sqrt{1+1/n^2}}$

As $n \to \infty$, $\frac{1}{n^2} \to 0$. Hence by L.L.,

$\frac{1}{\sqrt{1+1/n^2}} \longrightarrow \frac{1}{\sqrt{1+0}} = \frac{1}{1} = 1 > 0$

Since $\sum b_n$ divergent, by p-series, the series

2.

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$$ "looks like" (as $n \to \infty$) $\sum_{n=1}^{\infty} \frac{n}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Let $a_n = \dfrac{n-1}{n^2\sqrt{n}}$

$b_n = \dfrac{1}{n^{3/2}}$

Then,

$$\frac{a_n}{b_n} = \frac{n-1}{n^2\sqrt{n}} \cdot \frac{n^{3/2}}{1} = \frac{n\cancel{\sqrt{n}}(n-1)}{n^2\cancel{\sqrt{n}}} = \frac{1}{n}(n-1) = 1 - \frac{1}{n} \longrightarrow 1 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergent, by p-series test, we conclude $\sum a_n$ is convergent, by LCT.

3.

$$\sum_{n=1}^{\infty} \frac{9^n}{10^n - 4}$$ "Looks like" (as $n \to \infty$) $\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ (convergent geometric series)

Let $a_n = \dfrac{9^n}{10^n - 4}$

$b_n = \left(\dfrac{9}{10}\right)^n$

Then,

$$\frac{a_n}{b_n} = \frac{9^n}{10^n - 4} \cdot \left(\frac{10}{9}\right)^n = \frac{\cancel{9^n} \cdot 10^n}{(10^n - 4) \cdot \cancel{9^n}}$$

$$= \frac{10^n}{10^n - 4} = \frac{\cancel{10^n}}{\cancel{10^n}\left(1 - \frac{4}{10^n}\right)}$$

As $n \to \infty$, $\frac{4}{10^n} \to 0$ by G.P.T., $= \dfrac{1}{1 - 4/10^n} \longrightarrow \dfrac{1}{1-0} = 1 > 0$

Hence, $\sum a_n$ convergent, by LCT.

4. Explain why you can't use the Limit Comparison Test to determine convergence of

$$\sum_{n=1}^{\infty} \frac{2 - \sin(n)}{n^6 + n + 1}$$

by comparing with $\sum_n \frac{1}{n^6}$.

Let $a_n = \dfrac{2 - \sin(n)}{n^6 + n + 1}$; $\dfrac{a_n}{b_n} = \dfrac{2 - \sin(n)}{n^6 + n + 1} \cdot \dfrac{n^6}{1}$

$b_n = \dfrac{1}{n^6}$ $\quad = \dfrac{2 - \sin(n)}{\cancel{n^6}(1 + 1/n^5 + 1/n^6)} \cdot \dfrac{\cancel{n^6}}{1}$

As $n \to \infty$, $\frac{1}{n^5} + \frac{1}{n^6} \longrightarrow 0$, by L.L. P.T.O.

Hence, denominator converges to 1.

However, $2 - \sin(n)$ is not convergent as $n \to \infty$.

Appendix: In this Appendix we provide a proof of the Limit Comparison Test.

Let (a_n) and (b_n) be sequences of positive terms. Suppose that $\left(\frac{a_n}{b_n}\right)$ is convergent and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$. Hence, for $\epsilon = \frac{c}{2}$ we can find N so that

$$n \geq N \implies \left|\frac{a_n}{b_n} - c\right| < c_2$$

That is, for $n \geq N$ we have

$$\frac{c}{2} = c - \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} = c + \frac{c}{2}$$

Hence, since $b_n > 0$, we can rewrite this inequality as

$$\frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$$

Hence,

- if $\sum_{n=N}^{\infty} b_n$ is divergent so is $\sum_{n=N}^{\infty} a_n$, using the left inequality and DCT. Hence, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$ is also divergent.
- if $\sum_{n=N}^{\infty} b_n$ is convergent then so is $\sum_{n=N}^{\infty} a_n$, using the right inequality and DCT. Hence, $\sum_{n=1}^{\infty} a_n \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$ is also convergent.

Note: $\displaystyle\sum_{n=1}^{\infty} \frac{2 - \sin(n)}{n^6 + n + 1}$ can be compared with $\displaystyle\sum \frac{3}{n^6 + n + 1}$

Indeed: $-1 \leq \sin(n) \leq 1$

$\Rightarrow 1 \geq -\sin(n) \geq -1$

$\Rightarrow 3 \geq 2 - \sin(n) \geq 1$

Hence, $\displaystyle 0 < \frac{2 - \sin(n)}{n^6 + n + 1} \leq \frac{3}{n^6 + n + 1}$

The series $\displaystyle\sum \frac{3}{n^6 + n + 1}$ is convergent, by using LCT (compare with $\sum \frac{1}{n^6}$)

Hence, by DCT, $\displaystyle\sum_{n=1}^{\infty} \frac{2 - \sin(n)}{n^6 + n + 1}$ is convergent.