Note: many possible approaches to solutions!

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MATH 122 : HW SOL^N, ~~[illegible]~~ MAR. 2

1a) For each $n = 1, 2, 3, \ldots$

$$5n - 2 < 5n$$

$$\Rightarrow \quad \frac{1}{5n} < \frac{1}{5n-2}$$

$$\Rightarrow \quad \frac{\pi}{5n} < \frac{\pi}{5n-2}$$

Since $\sum_{n=1}^{\infty} \frac{\pi}{5n}$ divergent (by p-series), we find $\sum_{n=1}^{\infty} \frac{\pi}{5n-2}$ divergent, by DCT.

b) Note for each $n = 1, 2, 3 \ldots$

$$\sin(n) \leq 1$$

$$\Rightarrow \quad 2 + \sin(n) \leq 3$$

$$\Rightarrow \quad \frac{2 + \sin(n)}{n^3 + 2n + 1} \leq \frac{3}{n^3+2n+1} \leq \frac{3}{n^3}$$

Since $\sum_{n=1}^{\infty} 3/n^3$ convergent, by p-series, we find $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^3+2n+1}$ convergent, by DCT.

c) For any $n = 1, 2, 3, \dots$

$$\sqrt{n^5} < \sqrt{n^5 + 3n^2 + 10}$$

$$\Rightarrow \quad \frac{1}{\sqrt{n^5 + 3n^2 + 10}} < \frac{1}{n^{5/2}}$$

$$\Rightarrow \quad \frac{4}{\sqrt{n^5 + 3n^2 + 10}} < \frac{4}{n^{5/2}}$$

The series $\sum_{n=1}^{\infty} \frac{4}{n^{5/2}}$ is convergent, by p-series test, hence $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n^5+3n^2+10}}$ convergent by DCT

d) For each $n = 1, 2, 3, \dots$

$$\frac{1}{n^n} = \frac{1}{n \cdot n \cdot n \cdots n} \leq \frac{1}{n^2}$$

Indeed: $n = 1$: $\frac{1}{1} = 1$

$n \geq 2$: $\frac{1}{n^n} = \frac{1}{n \cdot n \dots n}$

$$= \frac{1}{n^2} \cdot \left[\frac{1}{n} \cdots \frac{1}{n} \right]$$

$$\leq \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent, by p-series, we have $\sum_{n=1}^{\infty} \frac{1}{n^n}$ convergent, by DCT.

e) For $n = 1, 2, 3, \ldots$

$$n^4 = (n^2)^2 < (n^2+1)^2$$

$$\Rightarrow \quad \frac{1}{(n^2+1)^2} < \frac{1}{n^4}$$

$$\Rightarrow \quad \frac{5n+2}{(n^2+1)^2} < \frac{5n+2}{n^4} = \frac{5}{n^3} + \frac{2}{n^4}.$$

Each of $\sum_{n=1}^{\infty} \frac{5}{n^3}$ and $\sum_{n=1}^{\infty} \frac{2}{n^4}$ convergent by p-series, hence so is $\sum_{n=1}^{\infty} \frac{5}{n^3} + \sum \frac{2}{n^4}$

$$= \sum_{n=1}^{\infty} \frac{5}{n^3} + \frac{2}{n^4}$$

$$= \sum_{n=1}^{\infty} \frac{5n+2}{n^4}$$

$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{5n+2}{(n^2+1)^2}$ convergent, by DCT.

2a) Telescoping series

i) $$\sum_{n=2}^{m} \frac{1}{n(n-1)} = \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$= \sum_{n=2}^{m} \frac{1}{n-1} - \sum_{n=2}^{m} \frac{1}{n}$$

$$= \left(1 + \cancel{\tfrac{1}{2}} + \cdots + \cancel{\tfrac{1}{m-1}} \right) - \left(\cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{3}} + \cdots + \frac{1}{m} \right)$$

$$= 1 - 1/m$$

ii) Hence, since $\lim_{m \to \infty} \sum_{n=2}^{m} \frac{1}{n(n-1)}$

$$= \lim_{m \to \infty} (1 - 1/m) = 1$$

we conclude $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges to 1.

b) Let $b_n = \frac{1}{n^2}$. Then,

$$b_n \leq a_n, \quad \text{for all } n$$

Indeed: $b_1 = a_1$

and, for $n \geq 2$,

$$n^2 - n < n^2$$

$$\Rightarrow \frac{1}{n^2} < \frac{1}{n^2 - n} = \frac{1}{n(n-1)}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by DCT.

3a) F – eg. $a_n = 1/n^2$

b) T – use DCT with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

c) F – $a_n = b_n = \frac{1}{n^{2/3}}$

d) T – (s_m) is (strictly) increasing and bounded above $\Rightarrow$ convergent by MBT.

4a)
$$H_9 = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{9}$$
$$> \frac{1}{10} + \frac{1}{10} + \dots + \frac{1}{10} = 9/10$$

b)
$$H_{99} = H_9 + \frac{1}{10} + \frac{1}{11} + \dots + \frac{1}{99}$$
$$> H_9 + \frac{1}{100} + \frac{1}{100} + \dots + \frac{1}{100}$$

$$> \frac{9}{10} + \frac{90}{100} = 2\left(\frac{9}{10}\right)$$

c) $$H_{999} = H_{99} + \frac{1}{100} + \frac{1}{101} + \cdots + \frac{1}{999}$$

$$> H_{99} + \frac{1}{1000} + \frac{1}{1000} + \cdots + \frac{1}{1000}$$

$$> 2\left(\frac{9}{10}\right) + \frac{900}{1000} = 3\left(\frac{9}{10}\right)$$

d) Since $10^k - 1 = \overbrace{99\ldots9}^{k\ 9s}$

$$H_{10^k-1} = H_{99\ldots9} = H_{10^{k-1}-1} + \frac{1}{10^{k-1}} + \frac{1}{10^{k-1}+1} + \cdots + \frac{1}{10^k-1}$$

$$> H_{10^{k-1}-1} + \underbrace{\frac{1}{10^k} + \frac{1}{10^k} + \cdots + \frac{1}{10^k}}_{\substack{(10^k-1)-(10^{k-1}-1) \\ = 10^k - 10^{k-1} = 9\times10^{k-1} \text{ terms}}}$$

$$= H_{10^{k-1}-1} + \frac{9\times10^{k-1}}{10^k}$$

$$= H_{10^{k-1}-1} + \frac{9}{10}$$

Similarly, $$H_{10^{k-1}-1} > H_{10^{k-2}-1} + \frac{9}{10}$$

$$H_{10^{k-2}-1} > H_{10^{k-3}-1} + 9/10$$

$$\vdots$$

$$H_{99} > H_9 + \frac{9}{10} > 2\left(\frac{9}{10}\right)$$

Hence,

$$H_{10^k-1} > H_{10^{k-1}-1} + \frac{9}{10}$$

$$> H_{10^{k-2}-1} + \frac{9}{10} + \frac{9}{10} = H_{10^{k-2}-1} + 2\cdot\left(\frac{9}{10}\right)$$

$$> H_{10^{k-3}-1} + \frac{9}{10} + \frac{9}{10} + \frac{9}{10} = H_{10^{k-3}-1} + 3\cdot\left(\frac{9}{10}\right)$$

$$\vdots$$

$$> H_{99} + \frac{9}{10} + \cdots + \frac{9}{10} = H_{99} + (k-2)\cdot\left(\frac{9}{10}\right)$$

$$> 2\left(\frac{9}{10}\right) + (k-2)\left(\frac{9}{10}\right) = k\cdot\left(\frac{9}{10}\right)$$

e) Since $k\left(\frac{9}{10}\right)$ unbounded as $k \to \infty$, we have H_{10^k-1} unbounded as $k \to \infty$.

Hence, (H_n) unbounded.