Calculus II: Spring 2018

Homework

Due March 2, 4pm

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Some thoughts and advice:

- You should expect to spend at least 1 – 2 hours on problem sets. A lot of practice problem-solving is essential to understand the material and skills covered in class. Be organised and do not leave problem sets until the last-minute. Instead, get a good start on the problems as soon as possible.
- When approaching a problem think about the following: *do you understand the words used to state the problem? what is the problem asking you to do? can you restate the problem in your own words? have you seen a similar problem worked out in class? is there a similar problem worked out in the textbook? what results/skills did you see in class that might be related to the problem?*

  If you are stuck for inspiration, use the course `piazza` forum (accessible via the course Canvas site), come to office hours, or send me an email. However, don't just ask for the solution - provide your thought process, the difficulties you are having, and ask a coherent question in complete English sentences. Remember the 3RA approach to asking questions outlined in the course syllabus.
- Form study groups - get together and work through problem sets. **This will make your life easier!** You can use `piazza` to arrange meet-ups. However, you must write your solutions *on your own* and *in your own words*.
- If you would like more practice then there are (hundreds of) problems in the supplementary course textbooks mentioned in the syllabus, or you can check out `khanacademy.org`.
- You **are not allowed** to use any additional resources. If you are concerned then please ask.

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1. Use the Direct Comparison Test to determine whether the following series are convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \frac{\pi}{5n-2}$

   (b) $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^3+2n+1}$

   (c) $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n^5+3n^2+10}}$

   (d) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

   (e) $\sum_{n=1}^{\infty} \frac{5n+2}{(1+n^2)^2}$

2. In this problem you will show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. In fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, though we will have to wait some time to see why this is true.

   (a) Consider the series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$.

      i. Show that $\sum_{n=2}^{m} \frac{1}{n(n-1)} = 1 - \frac{1}{m}$, for $m = 2, 3, 4, \ldots$.

      ii. Conclude that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges and determine its limit.

   (b) By comparing the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $\sum_{n=1}^{\infty} a_n$, where $a_1 = 1$, $a_n = \frac{1}{n(n-1)}$, $n \geq 2$, show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

3. True/False (no justification needed)

   (a) If $0 < a_n < \frac{1}{n}$, for each n, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent

(c) If $\sum a_n$ and $\sum b_n$ are both divergent then $\sum a_n b_n$ is divergent.

(d) If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms and the associated partial sums satisfy $s_m < 10^{100}$, for every m, then $\sum_{n=1}^{\infty} a_n$ is convergent.

4. In this problem you will give another proof that the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Denote by H_m the m^{th} partial sum of the Harmonic Series:

$$H_m = \sum_{n=1}^{m} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}$$

(a) Using the fact that there are 9 single digit natural numbers, show that

$$H_9 > \frac{9}{10}$$

(b) Using the fact that there are 90 double digit natural numbers, show that

$$H_{99} > \frac{9}{10} + \frac{90}{100} = 2\left(\frac{9}{10}\right)$$

(c) Use a similar argument as above to show that

$$H_{999} > 3\left(\frac{9}{10}\right)$$

(d) Show that, for any natural number k,

$$H_{10^k-1} > k\left(\frac{9}{10}\right)$$

(e) Deduce that the sequence (H_k) is unbounded and that the Harmonic Series is divergent.