Middlebury College

Calculus II: Spring 2018

Contact: gmelvin@middlebury.edu

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 6.1, 6.2*, 6.3*,6.6.
- *Calculus*, Spivak, 3rd Ed.: Section 18.

Keywords: chain rule, natural logarithm

The Natural Logarithm II & Inverse Trigonometric Functions

We saw yesterday that the exponential function $\exp(x)$ admits an inverse function $\exp^{-1}(x)$. We also determined that

$$\frac{d}{dx}\exp^{-1}(x) = \frac{1}{x} \tag{$*$}$$

A Fundamental Interlude

Let $f(x)$ be a function. An **antiderivative** of $f(x)$ is a differentiable function $F(x)$ satisfying

$$\frac{d}{dx}F(x) = f(x).$$

Proposition: *If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ then*

$$F(x) = G(x) + c,$$

for some constant c.

The most important Theorem you saw in Calculus I was an approach to determining the antiderivative of a continuous function.

> **Fundamental Theorem of Calculus**
>
> Let $f(x)$ be a continuous function defined on the closed interval $a \leq x \leq b$. Then, the function
>
> $$F(x) = \int_a^x f(u)du$$
>
> is an antiderivative of $f(x)$.

The natural logarithm

We can restate $(*)$ as follows:

> $\exp^{-1}(x)$ *is an antiderivative of* $g(x) = \frac{1}{x}$.

We define the **natural logarithm function** to be the function

$$\ln(x) = \int_1^x \frac{dt}{t} \quad x > 0$$

By the Fundamental Theorem of Calculus, $\ln(x)$ is an antiderivative of $g(x) = \frac{1}{x}$. Hence, the Proposition implies that there is a constant c so that

$$\exp^{-1}(x) = \underline{\ln(x) + c}.$$

Since $\exp(0) = \underline{1}$, we have $\underline{\exp^{-1}(1) = 0}$. Hence,

$$\underline{0 = \exp^{-1}(1) = \ln(1) + C =} 0 + C$$

Hence, $\Rightarrow \quad C = 0$

> The **natural logarithm function** $\ln(x)$ defined above *is* the inverse function $\exp^{-1}(x)$,
>
> $$\ln(x) = \exp^{-1}(x)$$

Remark:

1. You will show in Homework that the function $\ln(x)$ just defined as the inverse of $\exp(x)$ satisfies the expected *logarithm rules*:

   - $\ln(xy) = \ln(x) + \ln(y)$, for every $x, y > 0$.
   - $\ln(x^n) = n \ln(x)$, for every $x > 0$ and natural number n.

2. As the inverse function of $\exp(x)$, the following functional relationships hold:

   - $\exp(\ln(y)) = y$, for every $y > 0$,
   - $\ln(\exp(x)) = x$, for every x.

3. We have claimed that

   $$\exp(x) = e^x, \qquad \text{where } e = \exp(1) \text{ is Euler's number.}$$

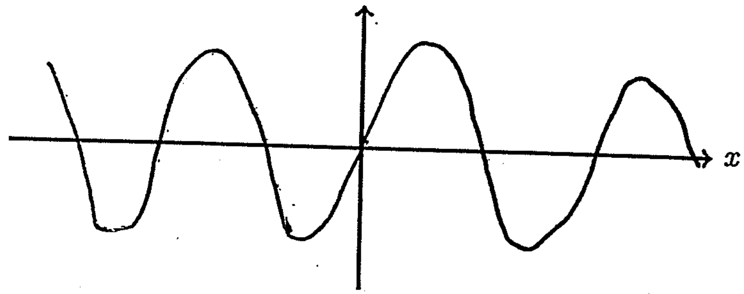
   Hence, we see that $\ln(x) = \log_e(x)$ is the *logarithm base e* function.

Inverse Trigonometric Functions

We are going to provide a similar analysis to determine **inverse trigonometric functions**.

CHECK YOUR UNDERSTANDING

1. Let $f(x) = \sin(x)$. Draw the graph of $f(x)$.



2. Explain why $f(x)$ is not one-to-one.

fails horizontal line test

(Recall: $f(x)$ is one-to-one if *distinct inputs give distinct outputs*)

3. Determine a domain $A: \; a \leq x \leq b$ on which $f(x)$ is one-to-one.

$-\frac{\pi}{2} \leq x \leq \pi/2$

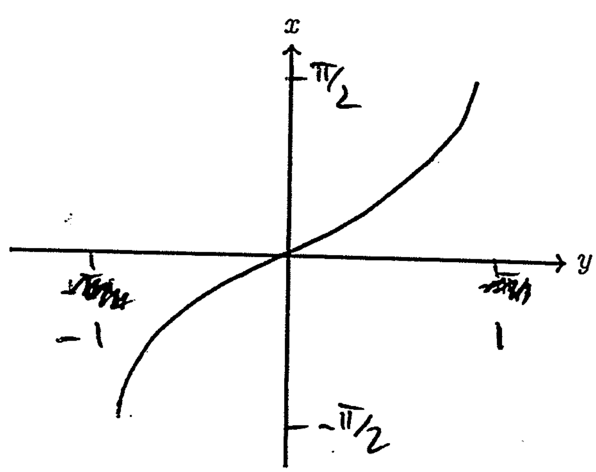
4. What is the range B of $f(x)$ when the inputs are restricted to A?

$[-1, 1]$

5. Explain why an inverse function $f^{-1}(y)$ to $f(x)$ exists, when we restrict to domain A.

one-to-one $\Rightarrow$ inverse exists

6. Draw the graph of $f^{-1}(y)$



Definition: Consider the function $f(x) = \sin(x)$, with domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Then, $f(x)$ is one-to-one and we call its inverse function $f^{-1}(y)$ the **inverse sine function**, which we denote $\arcsin(y)$.

CHECK YOUR UNDERSTANDING

Complete the following statement:

- the domain of $\arcsin(y)$ is $[-1, 1]$
- the range of $\arcsin(y)$ is $[-\pi/2, \pi/2]$

Remark:

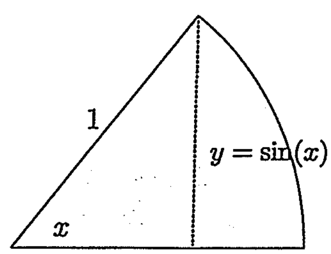
1. We write $\arcsin(y)$ instead of $\sin^{-1}(y)$ to avoid confusion with the common notation $\sin^k(x) = (\sin(x))^k$.
2. As the inverse function of $\sin(x)$, the following functional relationship holds:

   - $\sin(\arcsin(y)) = y$, for every $-1 \leq y \leq 1$
   - $\arcsin(\sin(x)) = x$, for every $-\pi/2 \leq x \leq \pi/2$

3. In words:

   "$\arcsin(y)$ is the arc whose sine is y"

   This is demonstrated by the following diagram: (recall that, the length of the arc drawn below is x, whenever the angle x is measured in radians)



Since $f(x) = \sin(x)$ is a differentiable function the same is true of $\arcsin(y)$. Using the formula for the derivative of an inverse function

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

we have

$$\boxed{\frac{d}{dy}\arcsin(y) = \frac{1}{\cos(\arcsin(y))} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-y^2}}}$$

Here we have used that the derivative of sin is cos, and used the above triangle to show that $\cos(\arcsin(y)) = \sqrt{1-y^2}$. Hence,

$$\boxed{\arcsin(x) \text{ is an antiderivative of } \frac{1}{\sqrt{1-x^2}}}$$

Exercise: determine the derivative of $\arctan(x)$ following a similar approach as above.