Middlebury College

# Calculus II: Spring 2018

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 6.1, 6.2*, 6.3*.
- *Calculus*, Spivak, 3rd Ed.: Section 18.

Keywords: chain rule, natural logarithm

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### The Natural Logarithm I

Today we introduce the *natural logarithm* function as the inverse function of $\exp(x)$.

Let $f(x)$ be a one-to-one function with domain $A$ and range $B$. Then, its **inverse function** $f^{-1}$ has domain $B$ and range $A$ and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad y = f(x).$$

In words,

| |
|---|
| If $y = f(x)$ is an output of $f$ then $y$ is an input of $f^{-1}$ and $f^{-1}(y) = x$. |

**Example:**

1. Let $f(x) = 1 - \frac{1}{x}$ with domain $A$ being the collection of all real numbers $x > 0$. The range of $f$, $B$, is the collection of all real numbers $y < 1$. The function $f(x)$ is one-to-one and its inverse function is
$$f^{-1}(y) = \frac{1}{1-y}, \quad y < 1.$$
2. Let $f(x) = 2x^2 - 1$, $x \geq 0$. Then, $f^{-1}(y) = \sqrt{\frac{y+1}{2}}$, $y \geq -1$.

## Important Remarks:

1. To determine the inverse function $f^{-1}(y)$ of a one-to-one function $f(x)$, solve the equation
$$y = f(x)$$
for $x$ in terms of $y$.
2. It is important to remember that $f^{-1}(y) \neq \frac{1}{f(y)}$, in general. For example, if $f(x) = 1 - \frac{1}{x}$ then
$$\frac{1}{f(y)} = \frac{y}{y-1} \neq \frac{1}{1-y} = f^{-1}(y)$$

3. Let $f(x)$ be a one-to-one function with domain $A$ and range $B$. Then, $f$ and its inverse function $f^{-1}$ satisfy the following functional relationship:

$$f(f^{-1}(y)) = y, \quad \text{for every } y \text{ in } B,$$
$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in } A.$$

## The derivative of inverse functions

Let $f(x)$ be a one-to-one function with domain $A$ and range $B$. Then, $f$ and its inverse function $f^{-1}$ satisfy the following functional relationship:

$$f(f^{-1}(y)) = y, \quad \text{for every } y \text{ in } B, \qquad (*)$$
$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in } A.$$

If $f(x)$ is also a differentiable function (i.e. the derivative $f'(x)$ exists for every $x$ in $A$) then its inverse function is also differentiable. In fact, the derivative of $f^{-1}(y)$ can be determined in terms of the derivative of $f'(x)$.

First, we recall some results from Calculus I.

CHECK YOUR UNDERSTANDING

Compute $\frac{dy}{dx}$ where

$$y = x^2 + 4x + \frac{1}{x^2 + 4x}$$

$$\frac{dy}{dx} = 2x + 4 - \frac{(2x+4)}{(x^2+4x)^2}$$

**Chain Rule**

Let $f$ and $g$ be differentiable functions. Then,

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

**Example:** Let $f(x) = x + \frac{1}{x}$ and $g(x) = x^2 + 4x$. We have

$$f'(x) = 1 - \frac{1}{x^2}, \qquad g'(x) = 2x + 4.$$

Then, the chain rule states that

$$\frac{d}{dx}\left(x^2 + 4x + \frac{1}{x^2 + 4x}\right) = \frac{d}{dx} f(g(x)) = f'(x^2 + 4x) \cdot g'(x) = \left(1 - \frac{1}{(x^2 + 4x)^2}\right)(2x + 4)$$

You can check that this agrees with your calculation above.

CHECK YOUR UNDERSTANDING

Recall the function $f(x) = 1 - \frac{1}{x}$ from October 11 Lecture. We determined the inverse function to be

$$f^{-1}(y) = \frac{1}{1-y}$$

1. Compute

$$\frac{d}{dy} f^{-1}(y).$$

$$= \frac{1}{(1-y)^2}$$

2. Compute $f'(x)$.

$$f'(x) = \frac{1}{x^2}$$

3. Show that

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}.$$

$$\frac{1}{f'(f^{-1}(y))} = \frac{1}{\frac{1}{\left(\frac{1}{1-y}\right)^2}} = \frac{1}{(1-y)^2}$$

**Derivative of the inverse function**

Let $f(x)$ be a differentiable one-to-one function. Suppose that $f'(f^{-1}(y)) \neq 0$, for all $y$. Then, $f^{-1}(y)$ is differentiable and

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

**Proof:** This follows from the functional relationship $(*)$ and the chain rule. We have, for every $y$,

$$f(f^{-1}(y)) = y.$$

Now, differentiating with respect to $y$ (remember, we are wanting the derivative of the function $f^{-1}(y)$ with respect to $y$) and using the chain rule, we find

$$1 = \frac{d}{dy}\left(f(f^{-1}(y))\right) = f'(f^{-1}(y)) \cdot (f^{-1})'(y)$$

$$\Longrightarrow \quad \frac{d}{dy}\left(f^{-1}(y)\right) = (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

## The natural logarithm

Recall the following facts about $\exp(x)$:

1. $\exp(x)$ is strictly increasing. Hence, $\exp(x)$ is one-to-one.
2. The domain of $\exp(x)$ is the collection of all real numbers.
3. The range of $\exp(x)$ is the collection of all $y > 0$.

Hence, $\exp(x)$ **has an inverse function** $\exp^{-1}(y)$.

**Remark:**

- the domain of $\exp^{-1}(y)$ is the collection of all $y > 0$
- the range of $\exp^{-1}(y)$ is the collection of all real numbers

We will often write $\exp^{-1}(x)$ instead of $\exp^{-1}(y)$. Remember, it doesn't matter what symbol we use for our input variable as long as we are consistent.

Now, since $f(x) = \exp(x)$ is a differentiable function so is $\exp^{-1}(y)$. Thus, using the formula for the derivative of the inverse function

$$\frac{d}{dy}\exp^{-1}(y) = \frac{1}{f'(\exp^{-1}(y))}$$

Recall that $f'(x) = \exp(x)$. Therefore, $f'(\exp^{-1}(y)) = \exp(\exp^{-1}(y)) = y$, using functional property $(*)$. Hence,

$$\boxed{\frac{d}{dy}\exp^{-1}(y) = \frac{1}{y} \qquad (**)}$$

## A Fundamental Interlude

Let $f(x)$ be a function. An **antiderivative** of $f(x)$ is a differentiable function $F(x)$ satisfying

$$\frac{d}{dx}F(x) = f(x).$$

**Proposition:** *If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ then*

$$F(x) = G(x) + c,$$

*for some constant $c$.*

The most important Theorem you saw in Calculus I was an approach to determining the antiderivative of a continuous function.